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Detecting alien limit cycles near a Hamiltonian 2-saddle cycle

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Abstract

This paper aims at providing an example of a cubic Hamiltonian 2-saddle cycle that after bifurcation can give rise to an alien limit cycle; this is a limit cycle that is not controlled by a zero of the related Abelian integral. To guarantee the existence of an alien limit cycle one can verify generic conditions on the Abelian integral and on the transition map associated to the connections of the 2-saddle cycle. In this paper, a general method is developed to compute the first and second derivative of the transition map along a connection between two saddles. Next, a concrete generic Hamiltonian 2-saddle cycle is analyzed using these formula's to verify the generic relation between the second order derivative of both transition maps, and a calculation of the Abelian integral.

Keywords: Planar vector field, Hamiltonian perturbation, limit cycle, Abelian integral, two-saddle cycle, alien limit cycle, transition map.

MSC2000: 34C23, 34C25, 34E10, 34C25, 34C20, 34C08

1 Introduction and settings

We deal with perturbations of Hamiltonian systems:

$$(X_{(\overline{\mu},\varepsilon)}):\begin{cases}\dot{x} &= -\frac{\partial H}{\partial y}+\varepsilon f,\\ \dot{y} &= \frac{\partial H}{\partial x}+\varepsilon g,\end{cases}\tag{1}$$

where $H(x,y), f(x,y,\overline{\mu},\varepsilon), g(x,y,\overline{\mu},\varepsilon)$ are C^∞ functions, ε is considered to take small positive values and $\overline{\mu}$ varies in some compact subset $K\subset\mathbb{R}^p$. Further we abbreviate $\mu=(\overline{\mu},\varepsilon)$.

We suppose that the flow of $X_{(\overline{\mu},0)}=X_H$ contains *a period annulus* bounded by a hyperbolic 2–saddle cycle $\mathcal{L}$ as in Figure 1. A period annulus is a subset of the plane filled by closed orbits of X_H. The hyperbolic 2–saddle cycle consists of two saddle–connections Γ_1 and Γ_2 and two hyperbolic saddles s_1 and s_2 such that $s_1:=\alpha(\Gamma_1)=\omega(\Gamma_2)$ and $s_2:=\alpha(\Gamma_2)=\omega(\Gamma_1)$. We choose H to be zero on the 2–saddle cycle and strictly positive on the nearby closed orbits.

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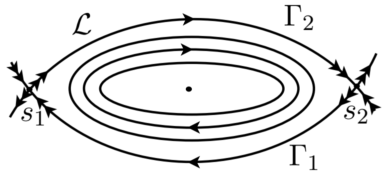

Figure 1: A 2–saddle cycle lying on the boundary of a period annulus.

In [9] it is proven that, for $\overline{\mu} \in K$ and $\varepsilon > 0$ near zero, $\mathcal{L}$ can produce limit cycles that are not controlled by zeros of the related Abelian integral (cfr. (5)); these limit cycles are also called '*alien limit cycles*' (cfr. [1]). In [9], one found that exactly one alien limit cycle exists in a '*generic*' unfolding (1) of codimension 4, leaving one connection of the 2-saddle cycle unbroken. A precise definition is given in Definition 2.

The principal result in this paper establishes the presence of this bifurcation phenomenon of alien limit cycles in the unfolding $(X_{(\overline{\mu},\varepsilon)})$ of the quadratic Hamiltonian system X_H with two centers and two heteroclinic loops:

$$\left\{ \begin{array}{lcl} \dot{x} & = & 1 - \frac{1}{4}y^2 - x^2 + \varepsilon[\overline{\mu}_3 xy + \overline{\mu}_4 y^2 x + y(x^2 + \frac{1}{12}y^2 - 1)(x - \frac{\sqrt{3}\pi}{8}xy)], \\ \dot{y} & = & 2xy + \varepsilon y(\overline{\mu}_1 + \overline{\mu}_2 x). \end{array} \right. \tag{2}$$

where the Hamiltonian H is given by

$$H(x,y) = y(x^2 + \frac{1}{12}y^2 - 1). \tag{3}$$

Notice that this system was also studied in [6]. Verification of the generic conditions that guarantee the presence of an alien limit cycle that bifurcates from the 2-saddle cycle lying in $\{y \leq 0\}$, is quite involved.

For an unfolding to be 'generic', besides a genericity property on the related Abelian integral (cfr. (5) , (6) , (9) and (10)), a genericity property of the second order derivative of the transition map along the saddle connections (cfr (7) , (8) and (11)) have to be satisfied as well. Sections 4, 5 and 6, present useful techniques to check the generic conditions in concrete examples.

In this paper we prove the following results. We obtain general formulas for the second order derivative of the transition map near a saddle connection, that remains unbroken in an smooth unfolding of a Hamiltonian vector field; these formulas are stated in section 6: Corollaries 18 and 19 respectively. Next, in section 7, using the developed machinery, in Section 7 the generic conditions are verified in the concrete system (2) ; we conclude that the generic properties described in Definition 2 all are satisfied; in particular, from the result in [9], we can conclude with:

Theorem 1 *Let $(X_{(\overline{\mu},\varepsilon)})$ be the unfolding of the Hamiltonian vector field X_H given in* (2) *with Hamiltonian H given in* (3) *; let $\mathcal{L}$ be the 2-saddle cycle with saddle points* $(-1,0)$ *and* $(1,0)$ *, lying in the half plane* $\{y \leq 0\}$ *. Then,*

1. $(X_{(\overline{\mu},\varepsilon)})$ *is a generic unfolding of codimension* 4*, leaving the connection* $\{y = 0\}$ *unbroken, in the sense of Definition 2.*

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2. *Hence, the bifurcation diagram of limit cycles bifurcating from $\mathcal{L}$ with respect to $(\overline{\mu}, \varepsilon)$, for $\|(\bar{\mu}, \varepsilon)\|$ sufficiently small, $\varepsilon > 0$, exhibits a swallow tail catastrophy.*

3. *In particular, there exists an alien limit cycle bifurcating from $\mathcal{L}$ for $(\bar{\mu}, \varepsilon) = (0, 0)$; i.e. there exist parameter values $(\bar{\mu}, \varepsilon)$ arbitrarily close to $(0, 0)$ such that $X_{(\overline{\mu}, \varepsilon)}$ has 4 limit cycles tending to $\mathcal{L}$ when $\|(\bar{\mu}, \varepsilon)\| \to 0$, while the Abelian integral has at most 3 zeroes near $h = 0$.*

In this paper, we only study the 2-saddle cycle in the half plane $\{y \leq 0\}$; one can study the 2-saddle cycle in the half plane $\{y \geq 0\}$ with the same reasoning.

Notice that the results obtained in sections 4, 5 (and 6) are valid for arbitrary analytic families of vector fields (perturbations of a Hamiltonian vector field); only in section 7, we work with the concrete Hamiltonian unfolding (2).

The paper is organised as follows. In section 2, the generic conditions are specified; in section 3, appropriate normal forms near the hyperbolic saddles s_1 and s_2 are given, that are used to calculate the second order derivative of the transition maps R_μ^1 and R_μ^2 in Sections 4, 5 and 6. In section 4, relying on [2], general formulas for the second order derivative of the transition map near a saddle connection, that remains unbroken in an smooth family of vector fields. In section 5 (respectively 6), these formulas are translated for smooth families of vector fields when expressed in normalizing coordinates (respectively for smooth unfoldings of a Hamiltonian vector field). Finally, using the developed machinery, in Section 7 the generic conditions are verified in the concrete Hamiltonian system (2).

2 Generic conditions

Throughout this article we suppose that (X_μ) is a smooth unfolding of a Hamiltonian vector field X_H like in (1) such that X_H admits a period annulus bounded by a 2-saddle cycle $\mathcal{L}$ like in Figure 1 and where μ varies in some neighbourhood of $(\overline{\mu}_0, 0)$, $\overline{\mu}_0 \in K$. After a translation in parameter space, one can always suppose that $\overline{\mu}_0 = 0$. Furthermore, we suppose that the connection Γ_2 remains unbroken by the unfolding.

In studying limit cycles bifurcating from such a 2-saddle cycle $\mathcal{L}$ for $\overline{\mu} \in K$ and $\varepsilon > 0$ near zero, it is convenient to consider the so-called difference map Δ between two sections transverse to $\mathcal{L}$ (see [9]). We here briefly recall its definition. Take transverse sections Σ_1 (respectively Σ_2) and Σ_3 (respectively Σ_4) near s_1 and s_2 respectively, transverse to Γ_2 (respectively Γ_1). Let u, v, z and w be regular parameters that parametrize $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4 respectively. In the respective parametrizations, $\Gamma_2 \cap \Sigma_1$ is represented by $u = 0$, $\Gamma_2 \cap \Sigma_3$ by $z = 0$, $\Gamma_1 \cap \Sigma_2$ by $v = 0$, and $\Gamma_2 \cap \Sigma_4$ by $w = 0$, see Figure 2.

Then, we consider the regular transition maps R_μ^2 from Σ_1 to Σ_3 along Γ_2 defined by the flow of $-X_{(\bar{\mu},\varepsilon)}$, and R_μ^1 from Σ_2 to Σ_4 along Γ_1, defined by the flow of $X_{(\bar{\mu},\varepsilon)}$. Let D_μ^1 (respectively D_μ^2) be the corner passages near the saddle s_1 (respectively s_2) defined by the flow of $X_{(\bar{\mu},\varepsilon)}$ (respectively $-X_{(\bar{\mu},\varepsilon)}$), see Figure 2. We suppose that all these transition maps are expressed in function of the chosen regular parameter on the sections Σ_i, $i = 1, \ldots, 4$. They are only locally defined: ε as well as the regular parameter u, v, z, w take on small positive values.

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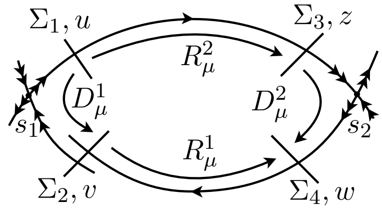

Figure 2: Transition maps near a 2–saddle cycle.

Now the difference map $\Delta_\mu : \Sigma_1 \to \Sigma_4$ is locally defined as:

$$\Delta_\mu(u) = \Delta(u, \mu) = \Delta^2(u, \mu) - \Delta^1(u, \mu),$$

for $u > 0$, with

$$\Delta^1(u, \mu) = \Delta^1_\mu(u) = R^1_\mu \circ D^1_\mu(u), \quad \Delta^2(u, \mu) = \Delta^2_\mu(u) = D^2_\mu \circ R^2_\mu(u).$$

Clearly, for μ near $(\overline{\mu}_0, 0), \overline{\mu}_0 \in K$, limit cycles of X_μ near $\mathcal{L}$, correspond to positive zeroes u of Δ_μ, for u near 0. In particular, we can write

$$\Delta = \varepsilon \overline{\Delta}, \tag{4}$$

for a C^∞ map $\bar{\Delta}$.

As for the traditional displacement map (retour mapping minus identity), the linear part of Δ with respect to ε is related to the Abelian integral $I_{\overline{\mu}}(h)$ for $\left(X_{(\bar{\mu},\varepsilon)}\right)_\varepsilon$ (cfr. [9]):

$$\overline{\Delta}(u, \overline{\mu}, 0) = I_{\overline{\mu}}(h) \equiv I(h, \overline{\mu}) \equiv \int_{\gamma_h} f dy - g dx, \qquad h > 0, \tag{5}$$

where γ_h is the non-isolated periodic orbit of X_H lying inside of $\{H = h\}$ and passing through the point u on Σ_1. Furthermore, it is well-known that $I(h, \overline{\mu})$ admits an asymptotic expansion in the logarithmic scale $1, h, \ldots, h^i, h^i \log h, \ldots$: there exist smooth functions p, q, r, s in $\bar{\mu}$ such that

$$I_{\overline{\mu}}(h) = p(\overline{\mu}) + q(\overline{\mu}) h \log h + r(\overline{\mu}) h + s(\overline{\mu}) h^2 \log h + O(h^2), \qquad h \downarrow 0. \tag{6}$$

The coefficients p, q, r, s in this expansion can be calculated using Picard-Fuchs equations (cfr. section 7 for an example).

The notion of a generic unfolding of codimension 4 also involves the asymptotics of the regular transition maps R^1_μ and R^2_μ. In Section 6, it will be shown that, up to terms of order $O(\varepsilon), \varepsilon \downarrow 0$, the transition maps R^1_μ and R^2_μ are the identity map, when expressed in appropriate normalizing coordinates near the saddles:

$$R^1_\mu(v) = v + \varepsilon(-\beta_1(\mu) + \gamma_1(\mu) v + \eta_1(\mu) v^2 + O(v^3)), u \downarrow 0, \tag{7}$$

for some smooth functions $\beta_1, \gamma_1, \eta_1$ in the parameter $\mu = (\overline{\mu}, \varepsilon)$ and

$$R^2_\mu(u) = u + \varepsilon(-\beta_2(\mu) + \gamma_2(\mu) u + \eta_2(\mu) u^2 + O(u^3)), u \downarrow 0,$$

for some smooth functions $\beta_2, \gamma_2, \eta_2$ in the parameter $\mu = (\overline{\mu}, \varepsilon)$. Furthermore, since Γ_2 remains unbroken, we have that $\beta_2(\mu) = 0, \forall\mu$, and after performing a parameter dependent coordinate change in u, one can suppose that $\gamma_2(\mu) = 0$ yielding to:

$$R^2_\mu(u) = u + \varepsilon(\eta_2(\mu)u^2 + O(u^3)), u \downarrow 0, \tag{8}$$

for some smooth function η_2 in $\mu = (\bar{\mu}, \varepsilon)$.

Definition 2 *Let $\left(X_{(\bar{\mu},\varepsilon)}\right)$ be a C^∞ unfolding of a Hamiltonian vector field X_H like in* (1). *Suppose that Γ is a 2-saddle cycle of X_H, of which one connection, say Γ_2, remains unbroken by the perturbation. Let $I_{\bar{\mu}}$ be the related Abelian integral of $\left(X_{(\bar{\mu},\varepsilon)}\right)_\varepsilon$ given in* (5) *with asymptotic expansion* (6). *Let $r_1(\mu)$ be the hyperbolicity ratio of the saddle of X_μ lying near s_1, then we define $\alpha_1(\overline{\mu}) := \alpha_1(\overline{\mu}, 0)$ by*

$$r_1(\mu) = 1 + \varepsilon\alpha_1(\mu).$$

Let R^1_μ (respectively R^2_μ) be the regular transitions along the connection Γ_1 (respectively Γ_2). Then, we say that $\left(X_{(\bar{\mu},\varepsilon)}\right)$ is a generic unfolding of X_H of codimension 4, if

1. *the Abelian integral is of codimension 3, i.e.,*

$$p(0) = q(0) = r(0) = 0, \qquad s(0) \neq 0. \tag{9}$$

and the map

$$(\mathbb{R}^p, 0) \to \left(\mathbb{R}^4, 0\right) : \overline{\mu} \mapsto (p(\overline{\mu}), q(\overline{\mu}), r(\overline{\mu}), \alpha_1(\overline{\mu})) \tag{10}$$

is a local submersion at zero.

2. *the functions η_1 and η_2, defined by the asymptotic expansions of R^2_μ and R^1_μ in* (7) *(respectively* (8)*), satisfy the following generic condition:*

$$\eta_2(0) \neq 2\eta_1(0). \tag{11}$$

Remark 3 *The main result in [9] implies that a 2-saddle cycle for such a generic Hamiltonian unfolding $\left(X_{(\bar{\mu},e)}\right)$ of codimension 4 can produce for limit cycles, while it is clear that the related Abelian integral $I_{\bar{\mu}}$ can have at most 3 zeroes that bifurcate from $h = 0$, for $\bar{\mu}$ near 0. This is striking since in case that $\mathcal{L}$ is a saddle-loop and the Abelian integral is generic, there is a 1-to-1 correspondence between the bifurcation diagram of the limit cycles perturbing from the Hamiltonian saddle loop and the zeroes of the related Abelian integral (cfr. [14]).*

Remark 4 *Notice that*

$$\eta_1(0) = \frac{1}{2}\frac{\partial^2 R^1_\varepsilon}{\partial v^2}(0) \text{ and } \eta_2(0) = \frac{1}{2}\frac{\partial^2 R^2_\varepsilon}{\partial u^2}(0).$$

The map R^1_μ describes the transition of the flow of $X_{(\bar{\mu},\varepsilon)}$ near a connection that is not preserved by the perturbation ($\varepsilon > 0$), therefore the calculation of $\eta_1(0)$ is more complicated than the one of $\eta_2(0)$. However, if there exists some $i_0 \in \{1, \ldots, p\}$ such that, for $p(\overline{\mu}) = I_{\overline{\mu}}(0)$,

$$\overline{p}(0) = 0 \quad \text{and} \quad \frac{\partial \overline{p}}{\partial \overline{\mu}_{i_0}}(0) \neq 0, \tag{12}$$

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which is guaranteed by conditions (9) *and* (10), *we can compute the quantity* $\eta_1(0)$ *using the formulas derived in section 6, that give expressions for the 1st and 2nd order derivative of a transition map near a connection that is preserved by the perturbation. Indeed, under the conditions in* (12), *there exists a subfamily* $(Z_\varepsilon)_\varepsilon$,

$$Z_\varepsilon = X_{(\gamma(\varepsilon),\varepsilon)},$$

induced by a smooth curve $\overline{\mu} = \gamma(\varepsilon), \varepsilon \downarrow 0,$ *in parameter space with* $\gamma(0) = 0$ *such that* Z_ε *has a 2-saddle cycle* $\mathcal{L}(\varepsilon)$ *for every* $\varepsilon > 0$ *sufficiently small (see for instance [11]). For* $\varepsilon = 0$, $\mathcal{L}(0) = \mathcal{L}$ *and, if we denote the connections of* $\mathcal{L}(\varepsilon)$ *by* $\Gamma_i(\varepsilon)$ *such that* α*-limit of* $\Gamma_i(\varepsilon)$ *for* Z_ε *is* $s_i, i = 1, 2,$ *then* $\Gamma_i(0) = \Gamma_i, i = 1, 2.$ *Denote the respective restrictions of the maps* $R^i_\mu, \eta_i, i = 1, 2$ *to the curve* $\bar{\mu} = \gamma(\varepsilon)$ *by* $\widetilde{R}^i_\varepsilon, \widetilde{\eta}_i, i = 1, 2,$ *we have*

$$\frac{d^2\tilde{R}^1_\varepsilon}{dv^2}(0) = 2\varepsilon\tilde{\eta}_1(\varepsilon), \quad \textit{and} \quad \frac{d^2\tilde{R}^2_\varepsilon}{du^2}(0) = 2\varepsilon\tilde{\eta}_2(\varepsilon);$$

in particular, since $\gamma(0) = 0$,

$$\eta_1(0) = \widetilde{\eta}_1(0) \;\; \textit{and} \; \eta_2(0) = \widetilde{\eta}_2(0).$$

3 Normal forms at hyperbolic saddles

In this section, we recall some useful normal forms for families of C^∞ vector fields near a hyperbolic saddle, that will simplify the calculations of the quantities $\eta_1(0)$ and $\eta_2(0)$; in particular, we give a specification of the isochore Morse lemma for a Hamiltonian unfolding in Theorem 10. Consider a C^∞ family of planar vector fields (X_μ) with parameter values μ varying in some open set $\mathcal{P}$ of $\mathbb{R}^p$. Suppose that for some $\mu_0 \in \mathcal{P}$, X_{μ_0} admits a hyperbolic saddle s, and suppose that the Jordan normal form of $DX_{\mu_0}(s)$ is given by

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

with $\lambda_2 < 0 < \lambda_1$. The *ratio of hyperbolicity of* X_{μ_0} *at* s is defined by $-\lambda_2/\lambda_1$.

As an easy consequence of the implicit function theorem, one can suppose that the saddle is persistant for all X_μ, $\mu \in \mathcal{P}$. By this we mean that there exists a C^∞ function $s : \mathcal{P} \mapsto \mathbb{R}^2$ such that each X_μ admits a hyperbolic saddle at $s_\mu := s(\mu)$ with $s_{\mu_0} = s$.

The following theorem can be found in [15].

Theorem 5 *Let* $(X_\mu)_{\mu\in\mathcal{P}}$ *be a* C^∞ *family as above such that* X_{μ_0} *admits a hyperbolic saddle* s*. Suppose that the ratio of hyperbolicity of* X_{μ_0} *at* s *is rational, given by* p/q *with* $p, q \in \mathbb{N}_1, (p, q) = 1$*. Then for each* $N \in \mathbb{N}$*, there exists a neighbourhood* $\mathcal{P}_N$ *of* μ_0 *in parameter space such that the* M*–jet,* $M = N(p + q) + 1$*, at* s_μ *of each* X_μ*,* $\mu \in \mathcal{P}_N$*, is locally* C^∞*-conjugate to:*

$$\tilde{X}^N_\mu : \begin{cases} \dot{x} &= x\left(\lambda_1 + \sum_{i=0}^N a_i(\mu)(x^p y^q)^i\right), \\ \dot{y} &= y\left(\lambda_2 + \sum_{i=0}^N b_i(\mu)(x^p y^q)^i\right), \end{cases} \tag{13}$$

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where the coefficients $a_i(\mu)$ and $b_i(\mu)$ are smooth in μ. In case the ratio of hyperbolicity of X_{μ_0} at s is irrational, then for every $N \in \mathbb{N}$, there exists a neighbourhood $\mathcal{P}_N$ of μ_0 in parameter space such that the N–jet of X_μ, $\mu \in \mathcal{P}_N$, is, locally near s_μ, C^∞ linearisable.

Remark 6 1. *The above theorem only applies to every finite jet of the family of vector fields, while for an individual vector field we have a normal form for its infinite jet at our disposal.*

2. *Using the theorem of Sternberg for families ([13]), it follows immediately from the above theorem that, in case the ratio of hyperbolicity of X_{μ_0} at s is rational, $\forall k \in \mathbb{N}$, there exists some $N(k) \geq k$ such that the family (X_μ), μ varying near μ_0, is, locally near s, C^k–conjugate to $\tilde{X}_\mu^{N(k)}$.*

In case of an individual integrable vector field a further simplification near a hyperbolic saddle can be obtained by applying Morse's lemma on the first integral H.

Proposition 7 *Let X be an integrable vector field with first integral H and admitting a hyperbolic saddle s. Denote by (u, v) the coordinates near s, given by Morse's lemma, in which H reads uv. Then, near s and expressed in the coordinates (u, v), X reads:*

$$\begin{cases} \dot{u} &= -u, \\ \dot{v} &= v, \end{cases} \tag{14}$$

up to C^∞ equivalence and a possible coordinate change in (u, v).

Proof. Denote by Y, the vector field, defined locally near the origin, that one obtains after expressing X in the coordinates (u, v). Because uv is a first integral of Y, it is clear that there exists a C^∞ function $\overline{Y}$, with $\overline{Y}(0) \neq 0$, such that $Y_1 = -u\overline{Y}$ and $Y_2 = v\overline{Y}$. After a possible coordinate switch in (u, v), one can always suppose that $\overline{Y}(0) > 0$ implying the desired result. ∎

We continue by describing another way to obtain the normal form (14), in case X is a Hamiltonian vector field. This method will first reduce X to a formal normal form (13) and can be useful when performing calculations in practice. However, using this method, the normal form (14) is only obtained on each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$, which will be sufficient for our further practical use of it.

We notice that similar reductions of Morse functions are already obtained in [12]. However these results were only valid near critical points that are not saddle points. The method that we propose is based on the following proposition.

Proposition 8 *Let X be an integrable vector field with first integral $H : V \subset \mathbb{R}^2 \to \mathbb{R}$ that admits a hyperbolic saddle s. Suppose that there exist C^∞ coordinates (u, v), near s, in which X reads:*

$$\begin{cases} \dot{u} &= -u, \\ \dot{v} &= v, \end{cases} \tag{15}$$

up to C^∞ equivalence. Then H, expressed in the coordinates (u,v), can be written as a C^∞ function in uv, locally near the origin, on each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$.

Proof. We prove the statement for H restricted to $\mathcal{H} = \{u \geq 0\}$; in an analogous way the statement is obtained for the other half planes. The orbits of system (15) are given by the curves $\{(0,0)\}$,

$$\{(u,v) : uv = c, u > 0, v > 0\}, \{(u,v) : uv = c, u > 0, v < 0\}, \text{ for } c \neq 0,$$
$$\{(0,v) : v > 0\} \text{ and } \{(0,v) : v < 0\}.$$

Choose c_0 such that $(c_0, 0) \in \mathcal{H} \cap V$, then we define the map f as follows: for c near zero, $f(c) = H(c_0, \frac{c}{c_0})$. By definition, the map f is C^∞. We now check that $f(uv) = H(u,v)$ on $\mathcal{H}$ and locally near the origin. As H is a first integral of system (15) on V, H stays constant on the orbits of (15) lying in $\mathcal{H} \cap V$.

Then for $u > 0$, this follows immediately from the definition of f. For $u = 0$, we notice that the fact that $f(0) = H(u_1, 0), \forall u_1 > 0$, the continuity of H implies $H(0,0) = f(0)$. Furthermore, because H stays constant on the positive and negative v–axis, it follows that $H(0,v) = H(0,0), \forall v$ implying the required result. ■

Proposition 9 *For a Hamiltonian vector field X_H, with* Hamiltonian H, *given by:*

$$\begin{cases} \dot{x} &= -\dfrac{\partial H}{\partial y}(x,y), \\ \dot{y} &= \dfrac{\partial H}{\partial x}(x,y), \end{cases} \tag{16}$$

and admitting a hyperbolic saddle s, there exist C^∞ coordinates (u,v), near s, in which the ∞–jet of X_H reads:

$$\begin{cases} \dot{u} &= -u\Big(\lambda + \sum_{i\geq 1} a_i (uv)^i\Big), \\ \dot{v} &= v\Big(\lambda + \sum_{i\geq 1} a_i (uv)^i\Big), \end{cases} \tag{17}$$

for some $a_i \in \mathbb{R}, i \in \mathbb{N}_1$.

Proof. Theorem 5 guarantees the existence of a C^∞ coordinate transformation $(x,y) = \varphi_1(u,v)$ near s in which the ∞–jet of X_H reads:

$$\begin{cases} \dot{u} &= -u\Big(\lambda + \sum_{i\geq 1} a_i (uv)^i\Big), \\ \dot{v} &= v\Big(\lambda + \sum_{i\geq 1} b_i (uv)^i\Big), \end{cases} \tag{18}$$

for some $\lambda > 0, a_i, b_i \in \mathbb{R}$, $i \in \mathbb{N}_1$. We prove that the coefficients a_i and b_i in (18) coincide. It is easily verified that the coordinate transformation $(x,y) = \varphi_1(u,v)$ transforms the Hamiltonian vector field into:

$$\begin{cases} \dot{u} &= -\dfrac{1}{\det D\varphi_1(u,v)} \dfrac{\partial H \circ \varphi_1}{\partial v}(u,v), \\ \dot{v} &= \dfrac{1}{\det D\varphi_1(u,v)} \dfrac{\partial H \circ \varphi_1}{\partial u}(u,v). \end{cases} \tag{19}$$

On the other hand using Borel's theorem on the realization of formal power series, one finds smooth functions f and g such that

$$j_\infty f(0)(z) = \sum_{i\geq 1} a_i z^i, \qquad j_\infty g(0)(z) = \sum_{i\geq 1} b_i z^i.$$

In particular φ_1 brings X_H into

$$\begin{cases} \dot{u} &= -u\Big(\lambda + f(uv)\Big) + R(u,v), \\ \dot{v} &= v\Big(\lambda + g(uv)\Big) + S(u,v), \end{cases} \tag{20}$$

with $j_\infty R(0) = j_\infty S(0) = 0$. After a suitable near–identity transformation, one can suppose that $R = S = 0$ [7]. Comparing (19) with (20), and abbreviating $\det D\varphi_1(u,v)$ as $D(u,v)$, one sees that

$$\begin{aligned} &\lambda\left(u\frac{\partial D}{\partial u}(u,v) - v\frac{\partial D}{\partial v}(u,v)\right) + u\frac{\partial D}{\partial u}(u,v)f(uv) - v\frac{\partial D}{\partial v}(u,v)g(uv) \\ &\qquad +D(u,v)\big(f(uv) - g(uv) + uv(f'(uv) - g'(uv))\big) = 0. \end{aligned} \tag{21}$$

It is easily seen that $\forall k \in \mathbb{N}_1$, the $2k$–jet at zero of the expression

$$u\frac{\partial D}{\partial u}(u,v) - v\frac{\partial D}{\partial v}(u,v) = 0,$$

does not contain terms in $(uv)^i, i \leq k$. Therefore comparing terms in uv of the 2–jet at zero of (21), one sees immediately that $a_1 = b_1$. By comparing terms in $(uv)^k$ in the $2k$–jet at $u = v = 0$ of (21), one can proceed by induction on $k \geq 1$ to prove that $a_k = b_k, \forall k \in \mathbb{N}_1$. ∎

The following theorem is a particular case of the 'isochore Morse lemma' proved by Colin de Verdières (for C^∞ vector fields on $\mathbb{R}^n, n \in \mathbb{N}$) in [3]; however, to keep the paper self-contained without being lengthy, we include here the theorem and its proof for C^∞ vector fields on $\mathbb{R}^2$.

Theorem 10 *Let X_H be a Hamiltonian vector field that admits a hyperbolic saddle s at which the eigenvalues of $DX_H(s)$ are given by $\pm\lambda, \lambda > 0$. Then there exist C^∞ coordinate coordinates $\varphi : C_1 \subset \mathbb{R}^2 \to \mathbb{R}^2, (x,y) = \varphi\ (n,m)$, where C_1 is a neighbourhood of s in $\mathbb{R}^2$ and a C^∞ function $d : C_2 \subset \mathbb{R} \to \mathbb{R}$, where C_2 is an open interval containing $0 \in \mathbb{R}$ $d(0) = 0$, such that X_H is transformed, up to the C^∞–equivalence factor $\lambda + d(nm)$, into*

$$\begin{cases} \dot{n} &= -n, \\ \dot{m} &= m, \end{cases} \tag{22}$$

In particular, if $\mathcal{H}$ is one of the half planes $\{n \geq 0\}$, $\{n \leq 0\}$, $\{m \geq 0\}$ or $\{m \leq 0\}$, then φ can be chosen such that the Hamiltonian $H\left(\varphi\left(n,m\right)\right) = nm$ on $\mathcal{H}$ and the equivalence factor equals $1/\det D\varphi(n,m) = \lambda + d(nm)$ on $\mathcal{H}$. Furthermore, we can suppose that $\{n = 1\}$ is contained inside of C_1.

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Proof. From Proposition 9 and using Borel's theorem on the realization of formal power series, one finds a C^∞ coordinate change $(x,y) = \varphi_1(u,v)$ bringing X_H, near s, into:

$$\begin{cases} \dot{u} &= -u(\lambda + f(uv)) + R(u,v), \\ \dot{v} &= v(\lambda + f(uv)) + S(u,v), \end{cases} \tag{23}$$

with f being smooth such that $j_\infty f(0)(z) = \sum_{i\geq 1} a_i z^i$ and R and S being flat at zero, i.e. $j_\infty R(0) = j_\infty S(0) = 0$. Applying a suitable near–identity transformation, one can suppose that the flat terms R and S are zero [7], such that the normal form of X_H for C^∞ equivalence reads $-u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$.

From Proposition 8, one knows that H, expressed in the new coordinates (u,v), is a function in uv locally near the origin on each of the half planes $\{u \geq 0\}$, $\{u \leq 0\}$, $\{v \geq 0\}$ or $\{v \leq 0\}$. Suppose $H(u,v) = uvH_0(uv)$ on $\{u \geq 0\}$ with $H_0(0) \neq 0$. After a reflection with respect to the u–axis, one can suppose that $H_0(0) > 0$. One performs the local transformation $(n,m) = \varphi_2(u,v)$, with

$$n = uG_0(uv), \qquad m = vG_0(uv), \tag{24}$$

where $G_0(uv) = \sqrt{H_0(uv)}$. This transformation will leave the linear normal form, up to C^∞ equivalence, invariant but will bring the Hamiltonian into nm on the half plane $\{n \geq 0\}$. In the new coordinates (n,m), X_H reads:

$$\begin{cases} \dot{n} &= -\dfrac{1}{\det D\varphi(n,m)} \dfrac{\partial H \circ \varphi}{\partial m}(n,m), \\ \dot{m} &= \dfrac{1}{\det D\varphi(n,m)} \dfrac{\partial H \circ \varphi}{\partial n}(n,m), \end{cases}$$

where $\varphi = \varphi_1 \circ \varphi_2^{-1}$. On the other hand it is a straightforward calculation to verify, using Proposition 8, that $f(uv)$ can be written as a function d in nm locally near the origin on the half plane $\{n \geq 0\}$. Therefore applying the transformation (24) on (23) (with $R = S = 0$), one finds:

$$\begin{cases} \dot{n} &= -n(\lambda + d(nm)), \\ \dot{m} &= m(\lambda + d(nm)), \end{cases}$$

for some C^∞ function d with $d(0) = 0$ implying the result on $\{n \geq 0\}$. The same arguments can be used for obtaining the result on the half planes $\{n \leq 0\}$, $\{m \geq 0\}$ or $\{m \leq 0\}$. Furthermore, by performing a dilatation, we can obtain that $\{n = 1\}$ is contained inside of C_1. ■

4 Transition maps

In this section we give formulas for the first and second derivative of the transition map, based on Diliberto's theorem.

Denote by X a C^∞ planar vector field with flow $\phi(t,v) := \phi_t(v), v \in \mathbb{R}^2$. Take two sections Σ_1 and Σ_2 transverse to some regular orbit of X. Suppose that $\psi_i = (f_i, g_i) : I_i \subset \mathbb{R} \mapsto \Sigma_i$ is a regular parametrization of Σ_i, for $i = 1, 2$. Denote by $T(s)$ the transition map of X from Σ_1 to Σ_2 expressed in the chosen parameters s and s'. In particular the orbit $\phi_t(\psi_1(s))$ crosses the section Σ_2 at the point $\psi_2(T(s))$. Let the function $\tau(s)$ be the transition time function of X from Σ_1 to Σ_2 expressed in the chosen parameter s; $\tau(s)$ is the time needed to go from $\psi_1(s)$ to $\psi_2(T(s))$.

Derivatives of the transition map and the transition time function can be found by means of implicit differentiation of the expression:

$$\theta(s, \tilde{s}, \tau(s)) = \phi_{\tau(s)}(\psi_1(s)) - \psi_2(\tilde{s}) = 0. \tag{25}$$

Denoting for fixed t, $D_2\phi(t, v)$ as the differential of $\phi(t, v)$ with respect to v, we find

$$D_2\phi(\tau(s), \psi_1(s))\, \psi_1'(s) + X(\phi_{\tau(s)}(\psi_1(s)))\tau'(s) - \psi_2'(T(s))T'(s) = 0. \tag{26}$$

To find the desired derivatives, we use Diliberto's theorem [4] to decompose the vectorial equation (26) with respect to an appropriate orthogonal basis. See also [2], where formulas for $T'(s)$ and $\tau'(s)$ are already obtained using Diliberto's theorem.

The scalar and wedge product between a vector field X with Euclidean coordinates (P, Q) and a vector field $\overline{X}$ with Euclidean coordinates $(\overline{P}, \overline{Q})$ are denoted as

$$X \cdot \overline{X} = P\overline{P} + Q\overline{Q}, \quad \text{and} \quad X \wedge \overline{X} := P\overline{Q} - Q\overline{P}.$$

Define the vector field

$$N := \frac{1}{\parallel X \parallel^2} X^{\perp},$$

multiple of the orthogonal vector field:

$$X^{\perp} = -Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y},$$

such that $X^{\perp} \cdot N = 1$. The following C^{∞} functions are referred to as the curl, the divergence and the curvature of X at p respectively:

$$\text{curl } X(p) = \frac{\partial Q}{\partial x}(p) - \frac{\partial P}{\partial y}(p), \qquad \text{div } X(p) = \frac{\partial P}{\partial x}(p) + \frac{\partial Q}{\partial y}(p),$$

and:

$$\kappa(p) = \frac{1}{\parallel X(p) \parallel} \left(N(p) \cdot \frac{d}{dt} X(\phi_t(p)) \mid_{t=0} \right). \tag{27}$$

Theorem 11 *(Diliberto [4]). Let X be a C^{∞} planar vector field with flow $\phi_t(v), v \in \mathbb{R}^2$. Let $p \in \mathbb{R}^2$ with $X(p) \neq 0$. For*

$$w = \alpha X(p) + \beta N(p)$$

the system:

$$\begin{cases} \dot{v} & = & DX(\phi_t(p))v, \\ v(0) & = & w \end{cases} \tag{28}$$

has solution

$$D_2\phi(t, p)w = A(t)X(\phi_t(p)) + B(t)N(\phi_t(p)),$$

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where $A(t) := A(t, X, p, w)$ *and* $B(t) := B(t, X, p, w)$ *are given by:*

$$A(t) = \alpha + \int_0^t \left\{ \frac{1}{\| X \|^2} [2\kappa \| X \| - \text{curl } X] \right\} (\phi_r(p)) \, B(r) \, dr, \tag{29}$$

$$B(t) = \beta \cdot \exp\left(\int_0^t \text{div } X(\phi_r(p)) \, dr \right).$$

Theorem 12 *Let* X *be a* C^∞ *vector field. Consider the transition map* $T(s)$ *between two sections* Σ_1 *and* Σ_2 *transverse to the flow of* X*. Suppose* ψ_1 *and* ψ_2 *are regular parametrisations of these sections and* Γ_s *is the orbit starting at* $\psi_1(s)$ *and ending in* $\psi_2(T(s))$*. Let the quantities* $\Delta_i(s), i = 1, 2,$ *be defined as:*

$$\Delta_i(s) := \Delta(s, X, \psi_i) = X(\psi_i(s)) \wedge \psi_i'(s),$$

and

$$\sigma_i(s) := \sigma(s, X, \psi_i) = \frac{\Delta_i'(s)}{\Delta_i(s)} - \frac{X(\psi_i(s)) \cdot \psi_i'(s)}{\| X(\psi_i(s)) \|^2} \text{div } X(\psi_i(s)),$$

with $i = 1, 2$*, the derivatives of first and second order of* T *are given by:*

$$T'(s) = \frac{\Delta_1(s)}{\Delta_2(T(s))} \exp \int_{\Gamma_s} \frac{\text{div } X}{\| X \|} \, d\overline{s}, \tag{30}$$

$$T''(s) = T'(s) \left(\sigma_1(s) - T'(s) \sigma_2(T(s)) + \Delta_1(s) \int_{\Gamma_s} \frac{\mathcal{A} \mathcal{B}}{\| X \|^3} \, d\overline{s} \right)$$

where $d\overline{s}$ *represents the arc length element of* Γ_s *and where* $\mathcal{A}(z) := \mathcal{A}(z, X)$ *and* $\mathcal{B}(z) := \mathcal{B}(z, X)$*,* $z = (x, y) \in \mathbb{R}^2$*, are given by:*

$$\begin{aligned} \mathcal{A}(z) &= D(\text{div } X)_z(X^\perp(z)) - \left\{ (2\kappa \| X \| - \text{curl } X) \, \text{div } X \right\}(z), \\ \mathcal{B}(z) &= \exp \int_{\Gamma_s(z)} \frac{\text{div } X}{\| X \|} \, d\overline{s}, \end{aligned} \tag{31}$$

with $\Gamma_s(z)$ *the orbit starting at* $\psi_1(s)$ *and ending in* z*.*

Proof. To shorten notation during the proof let us denote $\tilde{s} = T(s)$. We will decompose the vectorial equation (26) with respect to the orthogonal basis $\{X, N\}$ introduced in Theorem 11 to obtain formulas for $T'(s)$ and $\tau'(s)$.

Decomposing $\psi_i'(s)$ as

$$\psi_i'(s) = \alpha_i(s) X(\psi_i(s)) + \beta_i(s) N(\psi_i(s)) \tag{32}$$

with

$$\alpha_i(s) = \frac{X(\psi_i(s)) \cdot \psi_i'(s)}{\| X(\psi_i(s)) \|^2}, \qquad \beta_i(s) = X(\psi_i(s)) \wedge \psi_i'(s),$$

it follows from Theorem 11 that

$$D_2\phi(t, \psi_1(s))(\psi_1'(s)) = A(t) X(\phi_t(\psi_1(s)) + B(t) N(\phi_t(\psi_1(s))), \tag{33}$$

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where $A(t) = A(t, X, \psi_1(s), \psi_1'(s)$ and $B(t) = B(t, X, \psi_2(\tilde{s}), \psi_1'(s))$ are defined as in Theorem 11. This leads to the following decomposition of formula (26):

$$\begin{aligned}\Big(\alpha_2(\tilde{s})T'(s) - \tau'(s) - A(\tau(s))\Big)X(\psi_2(\tilde{s}))& \\ +\Big(\beta_2(\tilde{s})T'(s) - B(\tau(s))\Big)N(\psi_2(\tilde{s})) = 0.&\end{aligned} \tag{34}$$

In particular $\beta_2(\tilde{s})T'(s) - B(\tau(s)) = 0$ such that

$$T'(s) = \frac{\beta_1(s)}{\beta_2(\tilde{s})} \exp \int_0^{\tau(s)} \text{div } X(\gamma_s(t))\, dt \tag{35}$$

with $\gamma_s(t) = \phi(t, \psi_1(s))$. The first formula in (30) follows.

Derivation of (35) gives

$$\begin{aligned}T''(s) = T'(s)\Bigg(\frac{\beta_2(\tilde{s})}{\beta_1(s)}\frac{d}{ds}\left(\frac{\beta_1(s)}{\beta_2(\tilde{s})}\right) + \tau'(s)\,\text{div } X(\psi_2(\tilde{s}))& \\ + \int_0^{\tau(s)} \frac{d}{ds}\Big(\text{div } X(\gamma_s(t))\Big)\, dt\Bigg).&\end{aligned} \tag{36}$$

This formula can be simplified. To this end, we first search for an expression for $\frac{d}{ds}\,\text{div } X(\gamma_s(t))$. Because:

$$\frac{d}{ds}\Big(\text{div } X(\gamma_s(t))\Big) = D(\text{div } X)_{\gamma_s(t)}(D_2\phi(t, \psi_1(s))\psi_1'(s))$$

one finds, after substituting formula (33),

$$\begin{aligned}\frac{d}{ds}\Big(\text{div } X(\gamma_s(t))\Big) &= A(t)D(\text{div } X)_{\gamma_s(t)}(X(\gamma_s(t))) \\ &\quad +B(t)D(\text{div } X)_{\gamma_s(t)}(N(\gamma_s(t))).\end{aligned} \tag{37}$$

Since $D(\text{div } X)_{\gamma_s(t)}(X(\gamma_s(t))) = \frac{d}{dt}\,\text{div } X(\gamma_s(t))$ one can use the technique of partial integration on the integral

$$\int_0^{\tau(s)} A(t)D(\text{div } X)_{\gamma_s(t)}(X(\gamma_s(t)))\, dt.$$

Using (37) this yields

$$\int_0^{\tau(s)} \frac{d}{ds}\Big(\text{div } X(\gamma_s(t))\Big)dt = \Big[A(t)\,\text{div } X(\gamma_s(t))\Big]_0^{\tau(s)} + I, \tag{38}$$

where I is given by

$$\int_0^{\tau(s)} B(t)\frac{\mathcal{A}(\gamma_s(t))}{\| X(\gamma_s(t)) \|^2}\, dt = \beta_1(s) \int_{\Gamma_s} \frac{\mathcal{A}\mathcal{B}}{\| X \|^3}\, d\overline{s},$$

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with $\mathcal{A}$ and $\mathcal{B}$ defined as in (31). Substituting (38) and $\tau'(s) = \alpha_2(\tilde{s})T'(s) - A(\tau(s))$ (that follows from (34)) into (36) and using:

$$\frac{\beta_2(\tilde{s})}{\beta_1(s)}\frac{d}{ds}\left(\frac{\beta_1(s)}{\beta_2(\tilde{s})}\right) = \frac{\beta_1'(s)}{\beta_1(s)} - T'(s)\frac{\beta_2'(\tilde{s})}{\beta_2(\tilde{s})},$$

the formula of $T''(s)$ follows. ■

The formulas in the following corollary can already be found in [5]. One easily verifies that they are a special case of the formulas stated in Theorem 12.

Corollary 13 *Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be a C^∞ vector field. Suppose Γ is an orbit lying on the x-axis and $\Sigma_i = \{x = x_i\}$ are sections locally transverse to the flow of X at $(x_i, 0)$, $i = 1, 2$ and parametrised by $y \mapsto (x_i, y)$. Then the first two derivatives of the transition map T along Γ from Σ_1 to Σ_2 read:*

$$T'(0) = \exp\left(\int_{x_1}^{x_2} \frac{Q_y}{P}(x,0)dx\right),$$

$$T''(0) = T'(0)\int_{x_1}^{x_2} \frac{PQ_{yy} - 2P_yQ_y}{P^2}(x,0)\exp\left(\int_{x_1}^{x} \frac{Q_y}{P}(u,0)du\right)dx.$$

5 Transition along a saddle–connection

In this section, using the general formulas obtained in section 4, we derive formulas for the first and second derivative of the transition map along a 2-saddle connection, using normalizing coordinates near the saddles. In particular we don't restrict to individual vector fields but consider the transition near the 2-saddle connection in a family that leaves the saddle-connection unbroken.

We consider a C^∞ family of vector fields $(X_\mu)_{\mu\in\mathcal{P}}$ with parameter values μ varying in some subset $\mathcal{P} \subset \mathbb{R}^p$. We suppose that for $\mu_0 \in \mathcal{P}$, the vector field X_{μ_0} admits a saddle-connection Γ, with $\alpha(\Gamma) = s_1$ and $\omega(\Gamma) = s_2$, s_1 and s_2 hyperbolic saddles of X_{μ_0}. Let the vector field (X_μ) be expressed in the coordinates (x, y).

In particular we suppose that for μ near μ_0, X_μ has two hyperbolic saddles $s_1(\mu)$ and $s_2(\mu)$ lying in a neigbourhood of s_1 respectively s_2 such that $s_i(\mu_0) = s_i, i = 1, 2$ and that there exists a saddle-connection Γ_μ between them that coincides with Γ for $\mu = \mu_0$.

Let $i = 1, 2$. Denote the eigenvalues of the linear part $DX_{\mu_0}(s_i)$ at the saddle s_i as λ_i and ν_i with $\nu_i < 0 < \lambda_i$. Denote the ratio of hyperbolicity by $r_i = -\frac{\nu_i}{\lambda_i}$. Denote the eigenvalues of $DX_\mu(s_i(\mu))$ as $\lambda_i(\mu)$ and $\nu_i(\mu)$ with $\lambda_i(\mu_0) = \lambda_i$ and $\nu_i(\mu_0) = \nu_i$. The corresponding ratio of hyperbolicity of $s_i(\mu)$ is denoted as $r_i(\mu)$ and $r_i(\mu) = -\frac{\nu_i(\mu)}{\lambda_i(\mu)} = r_i + \tilde{r}_i(\mu)$, for some C^∞ function $\tilde{r}_i(\mu)$ with $\tilde{r}_i(\mu_0) = 0$.

Furthermore, we suppose that for μ near μ_0, the connection stays unbroken. This asumption is not restrictive by Remark 4.

Normalizing coordinates near the saddles. We suppose that $(X_\mu)_{\mu\in\mathcal{P}_0}$ can be brought into a normal form at both saddles s_1 and s_2. The normal form at s_i depends on the ratio of hyperbolicity r_i (see Theorem 5). From now on,

(n,m) will denote the normalizing coordinates near s_1 or s_2 depending on which saddle, s_1 or s_2, we are dealing with.

In case r_i is given by $p_i/q_i, p_i, q_i \in \mathbb{N}_1$, $(p_i, q_i) = 1$, there exist some C^k $(k \geq 2)$ coordinates, near the saddle s_i, in which the family $(X_\mu)_{\mu\in\mathcal{P}_0}$ reads:

$$\left\{\begin{array}{lcl} \dot{n} & = & n(\lambda_i(\mu) + a_i(\mu)n^{p_i}m^{q_i} + P_i(n^{p_i}m^{q_i}, \mu)), \\ \dot{m} & = & m(\nu_i(\mu) + b_i(\mu)n^{p_i}m^{q_i} + Q_i(n^{p_i}m^{q_i}, \mu)), \end{array}\right. \tag{39}$$

where $P_i(z,\mu)$ and $Q_i(z,\mu)$ are polynomials in $z = n^{p_i}m^{q_i}$ of degree $N(k) \geq k$ and of order $O(z^2)$. The functions λ_i, ν_i, a_i and b_i (respectively the polynomials P_i and Q_i) in (39) depend in a C^∞- (respectively C^k-) way on the parameter μ.

On the other hand if the ratio of hyperbolicity r_i of $DX_{\mu_0}(s_i)$ is irrational, then $(X_\mu)_{\mu\in\mathcal{P}_0}$ is C^k linearisable near the saddle s_i. In particular there exists some C^k coordinates near the saddle s_i in which $(X_\mu)_{\mu\in\mathcal{P}_0}$ reads:

$$\left\{\begin{array}{lcl} \dot{n} & = & \lambda_i(\mu)\, n, \\ \dot{m} & = & \nu_i(\mu)\, m. \end{array}\right. \tag{40}$$

The coordinate transformations expressing the coordinates (x,y) in function of (n,m) are denoted as φ^1_μ and φ^2_μ near s_1 and s_2 respectively. We choose normalizing coordinates near s_1 (resp. s_2) such that points on the positive $n-$axis correspond to points on the unstable (resp. stable) separatrix of s_1 (resp. s_2), lying on Γ for $\mu = \mu_0$. This can always be achieved by performing a suitable linear transformation in (n,m).

Let us denote the determinants of the corresponding jacobians of these transformations as $A^i_\mu(n,m) := \det D\varphi^i_\mu(n,m), i = 1, 2$. Further we also define

$$\theta^i_\mu(n,m) \quad = \quad \frac{\frac{\partial\varphi^i_\mu}{\partial n}(n,m).\frac{\partial\varphi^i_\mu}{\partial m}(n,m)}{\parallel \frac{\partial\varphi^i_\mu}{\partial n}(n,m) \parallel^2}. \tag{41}$$

Remark that geometrically $A^i_\mu(n,m)$ represents the area of the paralellogram spanned by the vectors $\frac{\partial\varphi^i_\mu}{\partial n}(n,m)$ and $\frac{\partial\varphi^i_\mu}{\partial m}(n,m)$. The angle between these two vectors is strongly related to the function $\theta^i_\mu(n,m)$.

Let us state the following lemma, which will be of use later on. It can be applied near both saddles s_1 and s_2 inside the family $(X_\mu)_{\mu\in\mathcal{P}_0}$.

Lemma 14 *Suppose (X_μ) is a family of vector fields such that X_{μ_0} admits a hyperbolic saddle $\overline{s}$ persisting as $\overline{s}(\mu)$ for μ near μ_0. Denote by (n,m) the normalizing coordinates in which the family is, near $\overline{s}$, expressed as the normal form $\overline{N}_\mu = \overline{N}^1_\mu \frac{\partial}{\partial n} + \overline{N}^2_\mu \frac{\partial}{\partial m}$ in (39) or (40). Let $(x,y) = \overline{\varphi}_\mu(n,m)$ be the corresponding C^k coordinate change. Denote by $\overline{\lambda}(\mu)$ and $\overline{\nu}(\mu)$ the eigenvalues of $DX_\mu(\overline{s}(\mu))$ with $\overline{\nu}(\mu) < 0 < \overline{\lambda}(\mu)$. Then*

$$X_\mu(\overline{\varphi}_\mu(n,m)) \wedge \frac{\partial\overline{\varphi}_\mu}{\partial m}(n,m) = \det D\overline{\varphi}_\mu(n,m)\overline{N}^1_\mu(n,m), \tag{42}$$

and

$$\overline{\lambda}(\mu)\, n\, \frac{X_\mu(\overline{\varphi}_\mu(n,0)).\frac{\partial\overline{\varphi}_\mu}{\partial m}(n,0)}{\parallel X_\mu(\overline{\varphi}_\mu(n,0)) \parallel^2} = \frac{\frac{\partial\overline{\varphi}_\mu}{\partial n}(n,0).\frac{\partial\overline{\varphi}_\mu}{\partial m}(n,0)}{\parallel \frac{\partial\overline{\varphi}_\mu}{\partial n}(n,0) \parallel^2}. \tag{43}$$

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Proof. The identities (42) and (43) can easily be deduced from the identity

$$X_\mu(\overline{\varphi}_\mu(n,m)) = D\overline{\varphi}_\mu(n,m)\overline{N}_\mu(n,m). \tag{44}$$

■

Expressing the transition using normalizing coordinates. Take a C^k normalizing coordinate transformation $(x,y) = \varphi^i_\mu(n,m)$ near s_1 and s_2 such that the normalizing coordinates transform the family (X_μ) into (N^i_μ) given by (22) , and satisfy the properties of theorem 10. In these normalizing coordinates (n,m) we choose sections $\Sigma^1_\mu = \{n = 1\}$ and $\Sigma^2_\mu = \{n = 1\}$ near s_1 and s_2 respectively that are transverse to the flow of the normal form (22) for $(X_\mu)_{\mu\in\mathcal{P}_0}$. In a natural way we use the normalizing coordinate m to parametrise the section.

The transition map from Σ^1_μ to Σ^2_μ expressed in the normalizing coordinate m is denoted as $R_\mu(m)$. Remark that $R_\mu(m)$ is only defined for m near zero and μ near μ_0. Calculating the derivatives of R_μ directly using Theorem 12 is not possible. Indeed only a finite jet of φ^1_μ and φ^2_μ at $(0,0)$ can be calculated implying that one is not able to calculate the derivatives of the parametrisations of the sections Σ^i_μ, $i = 1,2$. However this can be dealt with by using a limiting process (see also [8]).

Take $K_0 > 0$ and $\varepsilon_0 > 0$ such that $\{(x,y) \mid 0 \le n < K_0, -\varepsilon_0 < m < \varepsilon_0\}$ lies in the domains of φ^1_μ and φ^2_μ. For some $0 < K < K_0$ fixed, consider the section $C^i_{\mu,K} = \varphi^i_\mu(\{(K,m) \mid -\varepsilon_0 < m < \varepsilon_0\})$, parametrised by $\varphi^i_\mu \mid_{\{n=K\}} : m \mapsto \varphi^i_\mu(K,m)$, $i = 1,2$.

Consider the part of Γ_μ lying between the sections $C^1_{\mu,K}$ and $C^2_{\mu,K}$, denoted as $\Gamma_{\mu,K}$, and write $Z = \overline{T}_{\mu,K}(Y)$ as the transition map along $\Gamma_{\mu,K}$ from $C^1_{\mu,K}$ to $C^2_{\mu,K}$ expressed in the parameter m. Further let $F_{\mu,K}$ (respectively $G_{\mu,K}$) be the transition maps from $\{n = 1\}$ to $\{n = K\}$ near s_1 (respectively $\{n = K\}$ to $\{n = 1\}$ near s_2), expressed using as parameter the normalizing coordinate m. Then the transition map R_μ can be seen as the composition $R_\mu = G_{\mu,K} \circ \overline{T}_{\mu,K} \circ F_{\mu,K}$.

The first two derivatives of R_μ at zero are now given by

$$R'_\mu(0) = G'_{\mu,K}(0)\overline{T}'_{\mu,K}(0)F'_{\mu,K}(0), \tag{45}$$

and

$$\begin{aligned} R''_\mu(0) = G''_{\mu,K}(0) \left(\overline{T}'_{\mu,K}(0)\right)^2 \left(F'_{\mu,K}(0)\right)^2 + G'_{\mu,K}(0)\,\overline{T}''_{\mu,K}(0) \left(F'_{\mu,K}(0)\right)^2 \\ +G'_{\mu,K}(0)\,\overline{T}'_{\mu,K}(0)\,F''_{\mu,K}(0). \end{aligned} \tag{46}$$

Because these equalities hold for every $0 < K < K_0$, one can switch over to the limit for $K \to 0$ causing the chosen sections $C^1_{\mu,K}$ and $C^2_{\mu,K}$ to tend arbitrarily close to the saddles. This process will enable us to calculate the derivatives as stated in the following theorem.

Theorem 15 *Let (X_μ) be a C^∞ family admitting for each parameter two hyperbolic saddles $s_1(\mu)$ and $s_2(\mu)$ with a saddle–connection Γ_μ between them. Let R_μ be the transition map from Σ^1_μ to Σ^2_μ along Γ_μ expressed using normalizing coordinates. Consider the normal form at s_i, (39) or (40), and the corresponding coordinate transformation φ^i_μ, $i = 1,2$. Let $\Gamma_{\mu,K}$ be the part of Γ_μ starting* *at $\varphi_\mu^1(K,0)$ and ending in $\varphi_\mu^2(K,0)$. Then one has*

$$R'_\mu(0) = \frac{A_\mu^1(0,0)\,\lambda_1(\mu)}{A_\mu^2(0,0)\,\lambda_2(\mu)} \lim_{K\to 0}\left[K^{r_2(\mu)-r_1(\mu)}\exp\left(\int_{\Gamma_{\mu,K}} \frac{\operatorname{div} X_\mu}{\parallel X_\mu \parallel}\, d\bar{s}\right)\right], \quad (47)$$

where $d\bar{s}$ represents the arc length element of Γ_μ. Suppose that $R'_\mu(0)=1$, then

$$R''_\mu(0) = \lim_{K\to 0}\left[\, U_\mu(K) + I_\mu(K) + \lambda_1(\mu)K^{1-r_1(\mu)}A_\mu^1(K,0)\int_{\Gamma_{\mu,K}} \frac{\mathcal{A}_\mu\,\mathcal{B}_\mu}{\parallel X_\mu \parallel^3}\, d\bar{s}\right], \quad (48)$$

where $\mathcal{A}_\mu(p) := \mathcal{A}(p, X_\mu)$, $\mathcal{B}_\mu(p) := \mathcal{B}(p, X_\mu)$ are defined as in Theorem 12 and where $U_\mu(K)$ is the difference $U_\mu^1(K) - U_\mu^2(K)$ with

$$U_\mu^i(K) := K^{-r_i(\mu)}\left(\frac{\frac{\partial A_\mu^i}{\partial y}(K,0)}{A_\mu^i(K,0)} - \frac{\theta_\mu^i(K,0)}{\lambda_i(\mu)K}\operatorname{div} X_\mu(\varphi_\mu^i(K,0))\right), \quad i=1,2.$$

The function $I_\mu(K)$ disappears for $r_i \notin \mathbb{N}$. When $r_i \in \mathbb{N}$ it is given by the difference $I_\mu^1(K) - I_\mu^2(K)$ with

$$I_\mu^i(K) := \frac{a_i(\mu)}{\lambda_i(\mu)}K^{-\tilde{r}_i(\mu)} - 2\frac{b_i(\mu) + a_i(\mu)r_i(\mu)}{\lambda_i(\mu)}\omega\left(K, \tilde{r}_i(\mu)\right), \quad i=1,2,$$

where ω is the traditional compensator defined by

$$\omega(K,\alpha) = \begin{cases} \dfrac{K^{-\alpha}-1}{\alpha} \text{ for } \alpha \neq 0 \\ \ln K \text{ for } \alpha = 0 \end{cases}. \quad (49)$$

Remark 16 *In expressions (47), (48) as well as in (60), some terms tend to infinity, but the limit value of the expression in the right-hand side exists and is finite, since the limit of the left-hand term is well-defined for $K \to 0$ by definition. In fact, in (47) for instance, the logarithm of the expression in between the brackets $[\ldots]$ is given by*

$$(r_2(\mu) - r_1(\mu))\ln K + \left(\int_{\Gamma_{\mu,K}} \frac{\operatorname{div} X_\mu}{\parallel X_\mu \parallel}\, d\bar{s}\right);$$

since its limit for $K \to 0$ is finite, the integral is divergent and its principal part is given by

$$-\left(r_2(\mu) - r_1(\mu)\right)\ln K + o(1), K \to 0.$$

In fact, we are not interested in these principal parts but just in the finite quantities which remain after subtracting these (non-interesting) principal terms. In some sense, it is a question of method: we want to compute quantities which are trivially known to be finite ($R'_\mu(0), R''_\mu(0)$, and next $\eta_1(0), \eta_2(0)$) and the method is to apply expressions which diverge in terms of a parameter K and to retain some finite residue.

Proof. We will successively calculate all derivatives appearing in the right-hand side of formula (45). Because the equality holds for all $0 < K < K_0$, one can take the limit as $K \to 0$ to find the desired derivative $R'_\mu(0)$.

For calculating the derivatives $F'_{\mu,K}(0)$ and $G'_{\mu,K}(0)$, one can use the formulas of Corollary 13. One computes $F'_{\mu,K}(0) = K^{-r_1(\mu)}$ and $G'_{\mu,K}(0) = K^{r_2(\mu)}$ such that

$$R'_{\mu}(0) = K^{r_2(\mu)-r_1(\mu)}\,\overline{T}'_{\mu,K}(0). \tag{50}$$

The derivative $\overline{T}'_{\mu,K}(0)$ can be calculated using formula (30) in Theorem 12. From Lemma 14, it easily follows that

$$\overline{T}'_{\mu,K}(0) = \frac{A^1_{\mu}(K,0)\,\lambda_1(\mu)}{A^2_{\mu}(K,0)\,\lambda_2(\mu)}\,\exp\left(\int_{\Gamma_{\mu,K}} \frac{\text{div } X_{\mu}}{\parallel X_{\mu}\parallel} d\bar{s}\right). \tag{51}$$

One now substitutes (51) into (50) and takes the limit as $K \to 0$. Because the coordinate transformations φ^1_{μ} and φ^2_{μ} are locally diffeomorphisms, A^i_{μ}, $i = 1, 2$ stays away from zero for K near 0 implying (47).

Assuming that $R'_{\mu}(0) = 1$, i.e. $G'_{\mu,K}(0)\,\overline{T}'_{\mu,K}(0)\,F'_{\mu,K}(0) = 1$, equation (46) simplifies to

$$R''_{\mu}(0) = \frac{G''_{\mu,K}(0)}{G'^2_{\mu,K}(0)} + \frac{\overline{T}''_{\mu,K}(0)}{\overline{T}'_{\mu,K}(0)}\,F'_{\mu,K}(0) + \frac{F''_{\mu,K}(0)}{F'_{\mu,K}(0)}. \tag{52}$$

Again we calculate all ingredients of the right-hand side in this identity after which we let K tend to zero.

The term $F'_{\mu,K}(0)\,(\overline{T}''_{\mu,K}(0)/\overline{T}'_{\mu,K}(0))$ in (52) can be computed by use of Theorem 12. We define $\sigma^1_{\mu}(K) := \sigma_1(0, X_{\mu}, \varphi_{\mu}\mid_{v=K})$ and $\sigma^2_{\mu}(K) := \sigma_2(0, X_{\mu}, \psi_{\mu}\mid_{w=K}$), where σ_1 and σ_2 are defined as in Theorem 12. From equation (50) and the assumption that $R'_{\mu}(0) = 1$, it follows that $\overline{T}'_{\mu,K}(0) = K^{r_1(\mu)-r_2(\mu)}$. Because $F'_{\mu,K}(0) = K^{-r_1(\mu)}$, one finds, using Lemma 14:

$$\begin{aligned} F'_{\mu,K}(0)\frac{\overline{T}''_{\mu,K}(0)}{\overline{T}'_{\mu,K}(0)} \quad = \quad & \Bigg(K^{-r_1(\mu)}\sigma^1_{\mu}(K) - K^{-r_2(\mu))}\sigma^2_{\mu}(K) + \\ & + \lambda_1(\mu)K^{1-r_1(\mu)}A^1_{\mu}(K,0)\int_{\Gamma_{\mu,K}} \frac{\mathcal{A}_{\mu}\,\mathcal{B}_{\mu}}{\parallel X_{\mu}\parallel^3} d\bar{s}\Bigg). \end{aligned}$$

The expressions for the functions $\sigma^i_{\mu}(K)$ can be simplified by applying Lemma 14. When $r_1 \notin \mathbb{N}$, in particular when $q_1 > 1$, we find

$$\sigma^1_{\mu}(K) = \frac{\frac{\partial A^1_{\mu}}{\partial y}(K,0)}{A^1_{\mu}(K,0)} - \frac{\theta^1_{\mu}(K,0)}{\lambda_1(\mu)K}\,\text{div } X_{\mu}(\varphi^1_{\mu}(K,0),$$

while in the case where $r_1 \in \mathbb{N}$, we find

$$\sigma^1_{\mu}(K) = \frac{\frac{\partial A^1_{\mu}}{\partial y}(K,0)}{A^1_{\mu}(K,0)} - \frac{\theta^1_{\mu}(K,0)}{\lambda_1(\mu)K}\,\text{div } X_{\mu}(\varphi^1_{\mu}(0,K) + \frac{a_1(\mu)}{\lambda_1(\mu)}K^{r_1}.$$

Totally similar expressions are obtained for $\sigma^2_{\mu}(K)$.

For the expression $F''_{\mu,K}(0)/F'_{\mu,K}(0)$ we use Corollary 13. One calculates that for $r_1 \notin \mathbb{N}$ this quantity vanishes and that for $r_1 \in \mathbb{N}$:

$$\frac{F''_{\mu,K}(0)}{F'_{\mu,K}(0)} = 2\frac{b_1(\mu) + a_1(\mu)r_1(\mu)}{\lambda_1(\mu)}\int_1^K x^{-(1+\tilde{r}_1(\mu))}dx. \tag{53}$$

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Computations of the same sort also reveal an expression for $G''_{\mu,K}(0)/G'_{\mu,K}(0)$ when $r_2 \in \mathbb{N}$:

$$\frac{G''_{\mu,K}(0)}{G'^2_{\mu,K}} = -2\frac{b_2(\mu) + a_2(\mu)r_2(\mu)}{\lambda_2(\mu)} \int_1^K x^{-(1+\tilde{r}_2(\mu))} dx \tag{54}$$

By (49) , the integrals in (53) and (54) are the compensators $\omega(K, 1 + \tilde{r}_1(\mu))$ and $\omega(K, 1 + \tilde{r}_2(\mu))$ respectively. Doing the obvious substitutions into (52) and taking the limit as $K \to 0$ yields the formula for $R''_\mu(0)$ in (48). ■

Notice that in practice Theorem 15 can only be used when $R'_\mu(0) = 1$, which is true after a coordinate transformation $\tilde{m} = R'_\mu(0)m$ in normalizing coordinates near s_1.

6 Regular transition maps near a Hamiltonian 2–saddle cycle

In this section, we apply the formulas obtained in the previous section on an unfolding of a Hamiltonian vector field.

Consider a family (X_μ) like in (1), μ varying near $\mu_0 = (\overline{\mu}_0, 0)$ with $\overline{\mu}_0 \in \mathbb{R}^p$, and where X_H has a saddle–connection Γ on which the Hamiltonian takes constant value 0. Denote s_1 and s_2 as the hyperbolic saddles that are respectively given by the α–limit and ω–limit of Γ. Assume that Γ persists in the family (X_μ).

Appropriate normalizing coordinates near the saddles. Consider first the Hamiltonian vector field X_H near the saddles s_1 and s_2. Let $i = 1$ or $i = 2$. Denote the eigenvalues of $DX_H(s_i)$ as $\pm\lambda_i, \lambda_i > 0$. Theorem 10 guarantees the existence of coordinates $(\overline{n}, \overline{m})$ in which X_H, near s_i, reads:

$$\begin{cases} \dot{\overline{n}} &= \overline{n}, \\ \dot{\overline{m}} &= -\overline{m}, \end{cases} \tag{55}$$

up to a non–zero factor $E_i^0(\overline{n}, \overline{m})$ that equals $-\lambda_i$ for $\overline{n}\,\overline{m} = 0$. Denote by ψ_0^i the coordinate transformation expressing the old coordinates (x, y) in function of the new ones $(\overline{n}, \overline{m})$. Let $\mathcal{H}_i$ denote a half plane that contains, in its interior, the separatrices corresponding to the separatrices of s_i, that lie on $\mathcal{L}$ for $\varepsilon = 0$. One can choose ψ_0^i such that H expressed in the new coordinates reads $\overline{n}\,\overline{m}$ on $\mathcal{H}_i$ and such that $E_i^0(\overline{n}, \overline{m}) = -1/\det D\psi_0^i(\overline{n}, \overline{m})$ on $\mathcal{H}_i$.

Peforming, if necessary a coordinate switch or a reflection with respect to the origin, one can suppose that points on the positive $\overline{n}$–axes correspond to points on Γ_1 lying near s_1 and s_2 respectively; similar, This choice of coordinates implies an orientation on the normalizing coordinate axes such that $\det D\psi_0^1(0, 0) = 1/\lambda_1$ and $\det D\psi_0^2(0, 0) = -1/\lambda_2$. The half plane $\mathcal{H}_i$ can be chosen such that in the new coordinates it will correspond to $\{n \geq 0\}$.

We continue by peforming the transformation $(x, y) = \psi_0^i(\overline{n}, \overline{m})$ on the family (X_μ) yielding:

$$\begin{cases} \dot{\overline{n}} &= \overline{n} + \varepsilon\tilde{f}_i(\overline{n}, \overline{m}, \mu), \\ \dot{\overline{m}} &= -\overline{m} + \varepsilon\tilde{g}_i(\overline{n}, \overline{m}, \mu), \end{cases} \tag{56}$$

up to the factor $E_i^0(\overline{n}, \overline{m})$. Applying Theorem 5 yields C^k transformations $(\overline{n}, \overline{m}) = (I + \varepsilon\phi_\mu^i)(n, m)$ $(i = 1, 2)$, near s_i and for μ near μ_0, transforming (56) into the normal form:

$$\left\{ \begin{array}{rcl} \dot{n} & = & n\big(1 + \varepsilon\big(\tilde{\lambda}_i(\mu) + \tilde{a}_i(\mu)nm + P_i(nm, \mu)\big)\big), \\ \dot{m} & = & -m\big(1 + \varepsilon\big(\tilde{\nu}_i(\mu) - \tilde{b}_i(\mu)nm - Q_i(nm, \mu)\big)\big), \end{array} \right. \tag{57}$$

where $P_i(z, \mu)$ and $Q_i(z, \mu)$ are polynomials in $z = nm$ of finite degree $N(k) \geq k$ and of order $O(z^2)$.

Composing all the above performed transformations, one obtains C^k transformations $(x, y) = \varphi_\mu^1(n, m)$ and $(x, y) = \varphi_\mu^2(n, m)$ near the saddles s_1 and s_2 respectively and for μ near μ_0 bringing the family (X_μ) in (57) up to the factor $E_i(n, m, \mu) = \big(E_i^0 \circ (I + \varepsilon\phi_\mu^i)\big)(n, m)$, $i = 1, 2$.

Derivation of the transition along a Hamiltonian saddle–connection, expressed using appropriate normalizing coordinates. As before, choose transverse sections Σ_μ^1 and Σ_μ^2 corresponding to $\{n = 1\}$ in normalizing coordinates near s_1 and s_2 respectively. Similar choose transverse sections Σ_μ^2 and Σ_μ^4 corresponding to $\{m = 1\}$ in normalizing coordinates near s_1 and s_2 respectively. The sections Σ_μ^1 and Σ_μ^3 are parametrised using the normalizing coordinate m while the normalizing coordinate n is used in order to parametrize the sections Σ_μ^2 and Σ_μ^4.

Consider the transition maps $R_\mu^1(m)$ and $R_\mu^2(n)$ from Σ_1 to Σ_3 and Σ_2 to Σ_4 respectively, see Figure 2. We will derive formulas for the first and second derivative of $R_\mu := R_\mu^1$ at zero. Similar formulas for R_μ^2 can be deduced by applying a coordinate switch in normalizing coordinates.

Because we have chosen the normalizing coordinates in such way that for $\varepsilon = 0$ the Hamiltonian H reads nm in the normalizing coordinates near the saddles on $\{n \geq 0\}$, it is easily verified that $R_\mu = I + O(\varepsilon)$.

On the half plane $\{n \geq 0\}$, one can define for $i = 1, 2$:

$$\begin{array}{rcl} -E_i(n, m, \mu)\det\left(D\varphi_\mu^i(n, m)\right) & = & 1 + \varepsilon\overline{A}_{\overline{\mu}}^i(n, m) + O(\varepsilon^2), \\ \overline{\theta}_0^i(n, m) & = & \dfrac{\frac{\partial\psi_0^i}{\partial n}(n, m) \cdot \frac{\partial\psi_0^i}{\partial m}(n, m)}{\| \frac{\partial\psi_0^i}{\partial n}(n, m) \|^2}. \end{array} \tag{58}$$

Further, we let $\tilde{a}_i^0(\overline{\mu}) = \tilde{a}_i(\mu) \mid_{\varepsilon=0}$ and $\tilde{b}_i^0(\overline{\mu}) = \tilde{b}_i(\mu) \mid_{\varepsilon=0}$. We can now state the following proposition.

Proposition 17 *Suppose (X_μ) is a perturbation of a Hamiltonian vector field as in* (1)*, with μ varying in a neighbourhood of some $(\overline{\mu}_0, 0), \overline{\mu}_0 \in \mathbb{R}^p$, such that X_H admits a saddle–connection $\Gamma : H = 0$, between two hyperbolic saddles s_1 and s_2, that persists in the family (X_μ). Choose appropriate normalizing coordinates (n, m) near the saddles in which (X_μ) reads as in* (57) *and consider the functions $\overline{\theta}_0^i$ and $\overline{A}_\mu^i$ defined in* (58) *together with the coefficients $\tilde{a}_i^0(\overline{\mu})$ and $\tilde{b}_i^0(\overline{\mu})$.*

Consider the transition map from Σ_μ^1 to Σ_μ^3 expressed in the appropriate normalizing coordinates. Then we have

$$R'_\mu(0) = 1 + O(\varepsilon). \tag{59}$$

Denoting Γ_K as the part of Γ lying between $\psi_0^1(K,0)$ and $\psi_0^2(K,0)$ and $f_{\overline{\mu}}$, $g_{\overline{\mu}}$ as the restrictions of f and g to $\varepsilon = 0$, we have

$$R''_\mu(0) = \varepsilon \lim_{K\to 0}\left[\alpha(\overline{\mu}) + \delta(\overline{\mu})\ln K + \overline{U}_{\overline{\mu}}(K) - \int_{\Gamma_K} \frac{\overline{\mathcal{A}}}{\parallel X_H \parallel^3}\, ds\right] + O(\varepsilon^2), \quad (60)$$

where $\overline{\mathcal{A}}(z)$ is defined as

$$\overline{\mathcal{A}}(z) := D(\mathrm{div}\,(f_{\overline{\mu}}, g_{\overline{\mu}}))_z(X_H^{\perp}(z)) - \Big\{(2\kappa_0 \parallel X_H \parallel - \mathrm{curl}\ X_H)\,\mathrm{div}\,(f,g)\Big\}(z),$$

with $\kappa_0(z)$ the curvature of X_H at z, as defined in (27), and where $\overline{U}_{\overline{\mu}}(K)$ is given by the difference $\overline{U}^1_{\overline{\mu}}(K) - \overline{U}^2_{\overline{\mu}}(K)$ with

$$\overline{U}^i_{\overline{\mu}}(K) := \frac{1}{K}\left(\frac{\partial \overline{A}^i_{\overline{\mu}}}{\partial m}(K,0) + \frac{\overline{\theta}^i_0(K,0)}{\lambda_i\, K}\,\mathrm{div}\,(f,g)(\psi_0^i(K,0))\right), \quad i = 1,2.$$

The coefficient $\alpha(\overline{\mu})$ is given by $\tilde{a}_1^0(\overline{\mu}) - \tilde{a}_2^0(\overline{\mu})$ and $\delta(\overline{\mu})$ is given by $\delta_1(\overline{\mu}) - \delta_2(\overline{\mu})$ with $\delta_i(\overline{\mu}) = 2(\tilde{b}_i^0(\overline{\mu}) + \tilde{a}_i^0(\overline{\mu}))$.

Proof. We choose appropriate normalizing coordinates (n,m) near the saddles as before and apply Theorem 15. In the normalizing coordinates, all transitions occur in the half plane $\{n \geq 0\}$, even in $\{n \geq 0\} \cap \{m \geq 0\}$. So all calculations in normalizing coordinates can be restricted to the half plane $\{n \geq 0\}$.

Formula (59) is just the consequence of the fact that $R_\mu = I + O(\varepsilon)$. However it can also be seen by applying the formula (47). Let us explain how formula (60) follows from Theorem 15.

In the formulas of Lemma 14, we have to take the equivalence factors $E_i(n,m,\mu)$ into account, $i = 1,2$. Equation (42) stays valid up to this equivalence factor:

$$X_\mu(\varphi^i_\mu(n,m)) \wedge \frac{\partial \varphi^i_\mu}{\partial m}(n,m) = E_i(n,m,\mu)\det D\varphi^i_\mu(n,m) N^1_\mu(n,m), \qquad (61)$$

for $i = 1,2$ and where $N_\mu = N^1_\mu \frac{\partial}{\partial n} + N^2_\mu \frac{\partial}{\partial m}$ denotes the normal form (57) at s_1 or s_2 depending near which saddle we apply the identity. Equation (43) is translated into:

$$(1 + \varepsilon\tilde{\lambda}_i(\mu))\, nE_i(n,0,\mu)\,\frac{X_\mu(\varphi^i_\mu(n,0)).\frac{\partial \varphi^i_\mu}{\partial m}(n,0)}{\parallel X_\mu(\varphi^i_\mu(n,0)) \parallel^2} = \frac{\frac{\partial \varphi^i_\mu}{\partial n}(n,0).\frac{\partial \varphi^i_\mu}{\partial m}(n,0)}{\parallel \frac{\partial \varphi^i_\mu}{\partial u}(n,0) \parallel^2}. \quad (62)$$

Formula (61) implies that the area A^i_μ, in the proof of Theorem 15 is now replaced by:

$$\begin{aligned} A^i_\mu &= E_i(n,m,\mu)\det\left\{D\varphi^i_\mu(n,m)\right\} \\ &= -\big(1 + \varepsilon\overline{A}^i_\mu(n,m) + O(\varepsilon^2)\big), \quad i = 1,2. \end{aligned}$$

Further, for $\varepsilon = 0$, formula (62) leads to

$$-\lambda_i\, n\,\frac{X_H(\psi_0^i(n,0)).\frac{\partial \psi_0}{\partial m}(n,0)}{\parallel X_H(\psi_0^i(n,0)) \parallel^2} = \frac{\frac{\partial \psi_0^i}{\partial n}(n,0).\frac{\partial \psi_0^i}{\partial m}(n,0)}{\parallel \frac{\partial \psi_0^i}{\partial n}(n,0) \parallel^2} = \overline{\theta}^i_0(n,0), \qquad (63)$$

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such that $\theta^i_\mu(n,0)$ appearing in the formulas of Theorem 15 now equals $\overline{\theta}^i_0(n,0)+O(\varepsilon), \varepsilon \to 0$.

Noticing that the divergence of X_μ reads div $X_\mu = \varepsilon \,\mathrm{div}\,(f,g) + O(\varepsilon^2)$ and referring to the normal form in (57), it should be clear for the reader that formula (48) in Theorem 15 reduces to

$$R''_\mu(0) = \varepsilon r_1(K,\overline{\mu}) + \varepsilon^2 r_2(K,\mu), \tag{64}$$

where $r_1(K,\overline{\mu})$ is the function given by

$$r_1(K,\overline{\mu}) = \alpha(\overline{\mu}) + \delta(\overline{\mu})\ln K + U_\mu(K) - \int_{\Gamma_K} \frac{\overline{\mathcal{A}}}{\parallel X_H \parallel^3}\, ds,$$

with all appearing functions defined as above.

Notice that the transformation $\tilde{m} = R'_\mu(0)m$ in normalizing coordinates leaves the equality (64) invariant up to order $O(\varepsilon^2)$. Therefore, one can always assume that the condition $R'_\mu(0) = 1$ is satisfied such that it is justified to apply formula (48) for obtaining a formula for $R''_\mu(0)$ up to order $O(\varepsilon^2)$.

Because $R_\mu = I + O(\varepsilon)$, we can write:

$$R''_\mu(0) = \varepsilon\eta(\overline{\mu}) + O(\varepsilon^2), \tag{65}$$

for some function $\eta(\overline{\mu})$, C^∞ dependent on $\overline{\mu}$. In particular comparing (64) with (65), one sees $r_1(K,\overline{\mu}) = \eta(\overline{\mu})$, for all $0 < K < K_0$, K_0 near zero. This implies $\eta(\overline{\mu}) = \lim_{K\to 0} r_1(K,\overline{\mu})$, resulting in formula (60). ■

Formulas for calculating η_1 and η_2. Consider a family (X_μ) like in (1) containing a period annulus bounded by a hyperbolic 2–saddle cycle $\mathcal{L}$, see Figure 1, that leaves the connection Γ_2 unbroken. We choose H to be zero on the 2–saddle cycle and strictly positive on the nearby closed orbits. In the following corollaries, we obtain formulas for $\eta_1(\overline{\mu},0)$ (resp. $\eta_2(\overline{\mu},0)$), defined in (7) (resp. (8)) in the case where one can find a curve in parameter space passing through $(\overline{\mu},0)$ along which Γ_1 persists.

Corollary 18 *Suppose (X_μ) is a perturbation of a Hamiltonian vector field $X_H = X_{(\overline{\mu}_0,0)}$, containing a hyperbolic 2–saddle cycle $\mathcal{L}$, that leaves the connection Γ_2 unbroken. Suppose there exists a curve $\overline{\mu} = \gamma(\varepsilon)$ in parameter space passing through $(\overline{\mu}_0,0)$ along which the connection Γ_1 stays unbroken. Choose appropriate normalizing coordinates (n,m) near the saddles in which (X_μ) reads as in (57) and consider the functions $\overline{\theta}^i_0$ and $\overline{A}^i_\mu$ defined in (58) together with the coefficients $\tilde{a}^0_i(\overline{\mu})$ and $\tilde{b}^0_i(\overline{\mu})$. Let Γ^K_1 be the part of Γ_1 lying between $\psi^1_0(K,0)$ and $\psi^2_0(K,0)$. Denote f_{μ_0} and g_{μ_0} as the restrictions of f and g to $\mu = (\mu_0,0)$ respectively. Then the coefficient $\eta_1(\overline{\mu}_0,0)$ as defined in (7) reads*

$$\eta_1(\overline{\mu}_0,0) = \lim_{K\to 0}\left[\ \alpha(\overline{\mu}_0) + \delta(\overline{\mu}_0)\ln K + \overline{V}_{\overline{\mu}_0}(K) - \int_{\Gamma^1_K} \frac{\overline{\mathcal{A}}}{\parallel X_H \parallel^3}\, ds\right], \tag{66}$$

where $\overline{\mathcal{A}}(z)$ is defined as

$$\overline{\mathcal{A}}(z) := D(\mathrm{div}\,(f_{\mu_0},g_{\mu_0}))_z(X^\perp_H(z)) - \Big\{(2\kappa_0 \parallel X_H \parallel - \mathrm{curl}\ X_H)\,\mathrm{div}\,(f_{\mu_0},g_{\mu_0})\Big\}(z)$$

with $\kappa_0(z)$ *the curvature of* X_H *at* z*, defined as in* (27)*, and* $\overline{V}_{\overline{\mu}_0}(K)$ *is given by the difference* $\overline{V}^1_{\overline{\mu}_0}(K) - \overline{V}^2_{\overline{\mu}_0}(K)$ *with:*

$$\begin{aligned}\overline{V}^1_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\partial\overline{A}^2_{\overline{\mu}_0}}{\partial m}(K,0) - \frac{\partial\overline{A}^1_{\overline{\mu}_0}}{\partial m}(K,0)\right),\\ \overline{V}^2_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\overline{\theta}^1_0(K,0)}{\lambda_1 K}\,\mathrm{div}_1 + \frac{\overline{\theta}^2_0(K,0)}{\lambda_2 K}\,\mathrm{div}_2\right),\end{aligned}$$

where $\mathrm{div}_i := \mathrm{div}\,(f_{\mu_0}, g_{\mu_0})(\psi^i_0(K,0))$*. The coefficient* $\alpha(\overline{\mu}_0)$ *is given by* $\tilde{a}^0_2(\overline{\mu}_0) - \tilde{a}^0_1(\overline{\mu}_0)$ *and* $\delta(\overline{\mu}_0)$ *is given by* $\delta_2(\overline{\mu}_0) - \delta_1(\overline{\mu}_0)$ *with* $\delta^0_i(\overline{\mu}_0) = 2(\tilde{b}^0_i(\overline{\mu}_0) + \tilde{a}^0_i(\overline{\mu}_0))$*.*

Proof. Applying Proposition 17 on the family $(Z_\varepsilon) = (X_{(\gamma(\varepsilon),\varepsilon)})$, one easily obtains formula (66). ■

Corollary 19 *Suppose the same notations and considerations as in Corollary 18, and let* Γ^K_2 *be the part of* Γ_2 *lying between* $\psi^1_0(0,K)$ *and* $\psi^2_0(0,K)$*. Then, the coefficient* $\eta_2(\overline{\mu}_0, 0)$ *as defined in (8) reads:*

$$\eta_2(\overline{\mu}_0, 0) = \lim_{K\to 0}\left[\beta(\overline{\mu}_0) + \delta(\overline{\mu}_0)\ln K + \tilde{V}_{\mu_0}(K) - \int_{\Gamma^2_K}\frac{\overline{\mathcal{A}}}{\parallel X_H\parallel^3}\,ds\right], \tag{67}$$

where $\overline{\mathcal{A}}(z)$ *is defined as in Corollary 18 and* $\tilde{V}_{\overline{\mu}_0}(K)$ *is given by the difference* $\tilde{V}^2_{\overline{\mu}_0}(K) - \tilde{V}^1_{\overline{\mu}_0}(K)$*, with:*

$$\begin{aligned}\tilde{V}^1_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\partial\overline{A}^2_{\overline{\mu}_0}}{\partial n}(0,K) - \frac{\partial\overline{A}^1_{\overline{\mu}_0}}{\partial n}(0,K)\right),\\ \tilde{V}^2_{\overline{\mu}_0}(K) &:= \frac{1}{K}\left(\frac{\overline{\theta}^1_0(0,K)}{\lambda_1 K}\,\mathrm{div}_1 + \frac{\overline{\theta}^2_0(0,K)}{\lambda_2 K}\,\mathrm{div}_2\right),\end{aligned}$$

where $\mathrm{div}_i := \mathrm{div}\,(f_{\mu_0}, g_{\mu_0})(\psi^i_0(0,K))$*. The coefficient* $\beta(\overline{\mu}_0)$ *is given by* $\tilde{b}^0_2(\overline{\mu}) - \tilde{b}^0_1(\overline{\mu})$ *and* $\delta(\overline{\mu}_0)$ *is given by* $\delta_2(\overline{\mu}) - \delta_1(\overline{\mu})$ *with* $\delta^0_i(\overline{\mu}_0) = 2(\tilde{b}^0_i(\overline{\mu}_0) + \tilde{a}^0_i(\overline{\mu}_0))$*.*

Proof. After a coordinate switch $(n,m) \mapsto (m,n)$, we can apply Corrollary 18. The coefficients in the normal form (57) switch roles and change sign, if we want to keep the expression of (57) as it is. Because Γ_2 runs from s_1 to s_2, the roles of the saddles are interchanged compared with Corollary 18. ■

7 Unfolding a Hamiltonian 2–saddle cycle

In this section we verify that the unfolding $(X_{(\overline{\mu},\varepsilon)})$, $(\bar{\mu},\varepsilon) \sim (0,0)$, of the Hamiltonian vector field X_H, defined in (2) and (3) , satisfies the generic conditions in the sense of Definition 2.

The phase portrait of X_H contains four singularities: two centers at $(0,\pm 2)$ and two saddles given by $s_1 = (-1,0)$ and $s_2 = (1,0)$ where both saddles have eigenvalues ± 2 (cfr. Figure 3.). The singularities as well as the saddle-connection between them lying on the x–axis, remain fixed after perturbation. The saddles and the saddle-connection on the x-axis are part of two 2–saddle cycles, one lying in the half plane $\{y \geq 0\}$ and one lying in the half plane $\{y \leq 0\}$.

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Both 2–saddle cycles lie inside $\{H=0\}$; as in the announcement of theorem 1, we will only focus on the 2-saddle cycle that is contained in $\{H=0\}\cap\{y\le 0\}$, and call it by $\mathcal{L}$. The non-isolated periodic orbits inside $\mathcal{L}$, lie inside $\{H>0\}$.

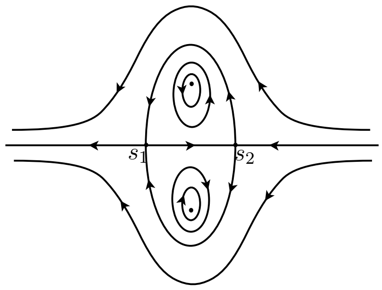

Figure 3: Phase portrait of $X_H = X_{(\overline{\mu},0)}$.

The conditions concerning the Abelian integral. Suppose $h \ge 0$ and denote γ_h as one of the closed curves inside the annulus of which $\mathcal{L}$ is the boundary. The Abelian integral is defined as:

$$I(h,\overline{\mu}) = \int_{\gamma_h} f dy - g dx, \tag{68}$$

with $f(x,y,\bar{\mu},\varepsilon)$ and $g(x,y,\bar{\mu},\varepsilon)$ the functions that appear after the parameter ε in the expression of (X_μ), (2). We now check the conditions (9) and (10).

In what follows, we calculate the coefficients in the expansion of I using Picard-Fuchs equations; write

$$I(h,\overline{\mu}) = p(\overline{\mu}) + q(\overline{\mu})h\log h + r(\overline{\mu})h + s(\overline{\mu})h^2\log h + O(h^2). \tag{69}$$

The Abelian integral related to (2) is given by:

$$\begin{aligned} I(h,\overline{\mu}) &= \int_{\gamma_h} \overline{\mu}_3 xydy - (\overline{\mu}_1 + \overline{\mu}_2 x)ydx \\ &\quad + \int_{\gamma_h} \overline{\mu}_4 y^2 xdy + \underbrace{y(x^2+\frac{1}{12}y^2-1)}_{=H(x,y)}(x - \tfrac{\sqrt{3}\pi}{8}yx)dy \\ &= \overline{\mu}_4 \int_{\gamma_h} y^2 xdy + \left(\overline{\mu}_3 - \tfrac{\sqrt{3}\pi}{8}h\right)\int_{\gamma_h} xydy \\ &\quad + h\int_{\gamma_h} xdy - \overline{\mu}_1\int_{\gamma_h} ydx - \overline{\mu}_2\int_{\gamma_h} xydx \end{aligned}$$

or

$$I(h,\overline{\mu}) = \overline{\mu}_4 I_2(h) + (\overline{\mu}_3 - \frac{\sqrt{3}\pi}{8}h)I_1(h) + (\overline{\mu}_1 + h)I_0(h), \tag{70}$$

with $I_k(h) = \int_{\gamma_h} y^k xdy$. Now by direct computation, one easily verifies that:

$$\lim_{h\to 0} I_0(h) = -\sqrt{3}\pi, \quad \lim_{h\to 0} I_1(h) = 8, \quad \lim_{h\to 0} I_2(h) = -3\sqrt{3}\pi.$$

In particular the condition to have a 2–saddle cycle is given by:

$$I(0,\overline{\mu}) = -3\sqrt{3}\pi\overline{\mu}_4 + 8\overline{\mu}_3 - \sqrt{3}\pi\overline{\mu}_1 = 0.$$

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Referring to [6], the Picard–Fuchs equation is given by:

$$D(h)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4}h^2-2 & -\frac{3}{4}h & \frac{2}{3} \\ h & \frac{9}{8}h^2 & -h \\ -\frac{3}{2}h^2 & -3h & \frac{3}{2}h^2 \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix}, \tag{71}$$

with $D(h) = \frac{9}{8}h(h^2 - \left(\frac{4}{3}\right)^2)$. Writing:

$$\begin{aligned} I_0(h) &= -\sqrt{3}\pi + a_1 h + a_2 h\log h + a_3 h^2 \log h + O(h^2), \\ I_1(h) &= 8 + b_1 h + b_2 h \log h + b_3 h^2 \log h + O(h^2), \\ I_2(h) &= -3\sqrt{3}\pi + c_1 h + c_2 h \log h + c_3 h^2 \log h + O(h^2), \end{aligned}$$

and substituting in (71) gives:

$$a_2 = -1, \qquad b_1 = -\sqrt{3}\pi, \qquad b_2 = 0, \qquad c_1 = 12, \qquad c_2 = 0.$$

We conclude that:

$$\begin{aligned} I_0(h) &= -\sqrt{3}\pi + a_1 h - h\log h + a_3 h^2 + a_4 h^2 \log h + O(h^2), \\ I_1(h) &= 8 - \sqrt{3}\pi h + b_3 h^2 + b_4 h^2 \log h + O(h^2), \\ I_2(h) &= -3\sqrt{3}\pi + 12h + c_3 h^2 + c_4 h^2 \log h + O(h^2). \end{aligned}$$

Using (70), the coefficients in (69) are given by:

$$\begin{aligned} p(\overline{\mu}) &= -3\sqrt{3}\pi\overline{\mu}_4 + 8\overline{\mu}_3 - \sqrt{3}\pi\overline{\mu}_1, \\ q(\overline{\mu}) &= -\overline{\mu}_1, \\ r(\overline{\mu}) &= 12\overline{\mu}_4 - \sqrt{3}\pi\overline{\mu}_3 + a_1\overline{\mu}_1, \\ s(\overline{\mu}) &= c_4\overline{\mu}_4 + b_4\overline{\mu}_3 + a_4\overline{\mu}_1 - 1, \end{aligned}$$

So $p(0) = q(0) = r(0) = 0$, but $s(0) \neq 0$. Moreover, since $\alpha_1(\overline{\mu}) = \frac{1}{2}(\overline{\mu}_1 - \overline{\mu}_2)$, it is easily seen that the map

$$\overline{\mu} \mapsto (p(\overline{\mu}), q(\overline{\mu}), r(\overline{\mu}), \alpha_1(\overline{\mu})),$$

is a local diffeomorphism at zero.

Calculation of appropriate normalizing coordinates. By the calculations of the Abelian integral and the observations in remark 4, we can use formulas (66) and (67) in order to calculate $\eta_2(0)$ as well as $\eta_1(0)$. Along $\{\overline{\mu} = 0\}$ the perturbation is zero on both connections of $\mathcal{L}$, implying that the 2-saddle cycle persist in the subfamily $(Z_\varepsilon)_\varepsilon$, where

$$Z_\varepsilon = X_{(\gamma(\varepsilon),\varepsilon)}, \text{ where } \gamma(\varepsilon) = 0, \forall \varepsilon \downarrow 0.$$

In what follows, notations are kept the same as in Corollaries 18 and 19.

We calculate the appropriate normalizing coordinates near the saddles of the subfamily $(X_\varepsilon) = (X_{(0,\varepsilon)})$. The unfolding (X_ε) reads:

$$(X_\varepsilon) : \begin{cases} \dot{x} &= 1 - \frac{1}{4}y^2 - x^2 + \varepsilon y(x^2 + \frac{1}{12}y^2 - 1)(x - \frac{\sqrt{3}\pi}{8}xy), \\ \dot{y} &= 2xy \end{cases}$$

with $X_0 = X_H$ and H defined in (3). It will appear to be sufficient to perform normal form calculations up to order 4.

The Hamiltonian vector field X_H has two hyperbolic saddles, one at $(-1,0)$ and one at $(1,0)$. Near $(1,0)$, we proceed as follows. We start with calculating the 3–jet of $(x,y) = \psi_0^2(\overline{n},\overline{m})$, the coordinate change transforming X_H into the normal form (55) up to C^∞ equivalence. First, we translate the singularity to the origin, yielding:

$$\begin{cases} \dot{x} &= -2x - \frac{1}{4}y^2 - x^2, \\ \dot{y} &= 2y + 2xy. \end{cases}$$

The linear part at the origin is already in its Jordan form. The transformation

$$(x,y) = (\overline{x} + \frac{1}{4}\overline{x}^3 + \frac{7}{48}\overline{xy}^2, \overline{y} + \frac{1}{4}\overline{x}^2\overline{y} - \frac{1}{48}\overline{y}^3), \tag{72}$$

will remove all terms of order less than 4. One concludes that the 3–jet of X is C^ω linearisable by the transformation given by the composition of the translation $(x,y) = (\overline{x}+1, \overline{y})$ with (72).

The Hamiltonian expressed in the new coordinates already reads $2xy$ up to order 5. Therefore the transformation that brings the Hamiltonian in xy is up to order 4 given by a dilatation that one can choose to be $(x,y) = (\overline{x}, \frac{\overline{y}}{2})$. After a switch of the normalizing coordinates and a reflection $(x,y) \mapsto (-x,-y)$, the positive x and y–axis in normalizing coordinates correspond respectively to the unstable and the stable separatrix of s_2 lying on $\mathcal{L}$. One obtains the following 3–jet of the transformation $(x,y) = \psi_0^2(\overline{n},\overline{m})$:

$$(x,y) = \Big(1 - \overline{m} + \tfrac{1}{2}\overline{m}^2 - \tfrac{1}{96}\overline{n}^2 - \tfrac{1}{4}\overline{m}^3 - \tfrac{7}{192}\overline{n}^2\overline{m}, \\ -\tfrac{1}{2}\overline{n} - \tfrac{1}{2}\overline{n}\,\overline{m} - \tfrac{1}{8}\overline{n}\,\overline{m}^2 + \tfrac{1}{384}\overline{n}^3\Big). \tag{73}$$

The 3–jet of (X_ε) is transformed into:

$$\begin{cases} \dot{\overline{n}} &= \overline{n} + \frac{1}{2}\varepsilon\overline{n}^2\overline{m}, \\ \dot{\overline{m}} &= -\overline{m} - \frac{1}{2}\varepsilon\overline{n}\,\overline{m} - \varepsilon\frac{\sqrt{3}}{32}\pi\overline{n}^2\overline{m}, \end{cases} \tag{74}$$

up to a factor $2 + O(|\overline{n}\,\overline{m}|^2)$.

Near $s_1 = (-1,0)$, one can make use of the symmetry of X_H with respect to the y–axis. The Hamiltonian vector field is invariant under the transformation

$$(x,y,t) \mapsto (-x,y,-t),$$

such that the behaviour of X_H in the region $\{(x,y) \mid -1 < x < -1+\varepsilon_0\}$ is exactly given by the behaviour of $-X_H$ in the region $\{(x,y) \mid 1-\varepsilon_0 < x < 1\}$. Choosing $\psi_0^1 := S \circ \psi_0^2$, where $S(x,y) = (-x,y)$, the 3–jet of (X_ε) is near s_1 transformed into:

$$\begin{cases} \dot{\overline{n}} &= \overline{n} - \frac{1}{2}\varepsilon\overline{n}^2\overline{m}, \\ \dot{\overline{m}} &= -\overline{m} + \frac{1}{2}\varepsilon\overline{n}\,\overline{m} + \varepsilon\frac{\sqrt{3}}{32}\pi\overline{n}^2\overline{m}, \end{cases} \tag{75}$$

up to a factor $-2 + O(|\overline{n}\,\overline{m}|^2)$. Moreover the Hamiltonian expressed in new coordinates reads $\overline{n}\,\overline{m}$, up to order 5.

One continues by performing a transformation of the form $I + \varepsilon\varphi_\varepsilon^i$, $i = 1, 2$ near s_1 and s_2 respectively, keeping the unperturbed vector field unchanged but removing non–resonant terms of order less than 4, appearing after the parameter ε in expressions (74) and (75).

Performing the transformation

$$(\overline{n}, \overline{m}) = \left(n, m - \varepsilon(\frac{1}{2}nm + \left(\frac{1}{8}\varepsilon - \frac{\sqrt{3}\pi}{64}\right) n^2 m)\right),$$

one comes to the following normal form at s_2 for the 3–jet of (X_ε):

$$\begin{cases} \dot{n} &= n + \frac{1}{2}\varepsilon n^2 m, \\ \dot{m} &= -m, \end{cases} \tag{76}$$

up to a factor $2 + O(|\overline{n}\,\overline{m}|^2)$.

Analogously, performing the transformation

$$(\overline{n}, \overline{m}) = \left(n, m + \varepsilon(\frac{1}{2}nm + \left(\frac{1}{8}\varepsilon + \frac{\sqrt{3}\pi}{64}\right) n^2 m)\right),$$

the 3–jet of (75) will, locally near s_1, be transformed into:

$$\begin{cases} \dot{n} &= n - \frac{1}{2}\varepsilon n^2 m, \\ \dot{m} &= -m, \end{cases} \tag{77}$$

up to a factor $-2 + O(|\overline{n}\,\overline{m}|^2)$.

Calculation of $\eta_1(0)$ and $\eta_2(0)$. We use formulas (66) and (67) to calculate $\eta_1(0)$ and $\eta_2(0)$. Using the above normal form calculations, one computes the functions $\overline{\theta}_0^i$ and $\overline{A}_0^i, i = 1, 2$, defined in (58) :

$$\overline{\theta}_0^1(n, m) = \overline{\theta}_0^2(n, m) = \frac{13}{12}x - \frac{11}{24}xy + O(\| (x, y) \|^3),$$

where x and y depend on (n, m) and:

$$\begin{cases} \overline{A}_0^1(n, m) &= \frac{1}{2}n + \frac{1}{64}\sqrt{3}\pi n^2 + O(\| (n, m) \|^3), \\ \overline{A}_0^2(n, m) &= -\frac{1}{2}n - \frac{1}{64}\sqrt{3}\pi n^2 + O(\| (n, m) \|^3), \end{cases}$$

together with:

$$\operatorname{div}(f_0, g_0)(x, y) = 2xy(x - \frac{\sqrt{3}\pi}{8}xy) + H(x, y)(1 - \frac{\sqrt{3}\pi}{8}y), \tag{78}$$

where f_0 and g_0 are the functions appearing after the parameter ε in the expression of (X_ε). In particular, one gets:

$$\operatorname{div}(f_0, g_0)(\psi_0^i(0, K)) = 0, \quad i = 1, 2, \quad \forall 0 < K < K_0,$$

such that $\tilde{V}_0^2(K) = 0$, for each K near zero, in formula (67). On the other hand, the asymptotic behaviour of $\tilde{V}_0^1(K)$ as $K \to 0$ is given by:

$$\tilde{V}_0^1(K) = -\frac{1}{K} + O(K), \quad K \to 0.$$

One concludes that the function $\tilde{V}_0(K)$ in (67) has the following asymptotic behaviour as $K \to 0$:

$$\tilde{V}_0(K) = \frac{1}{K} + O(K), \quad K \to 0. \tag{79}$$

Furthermore, from (41), we have:

$$\overline{\theta}_0^1(K,0) = \overline{\theta}_0^2(K,0) = \frac{13}{12}K + O(K^3), \quad K \to 0$$

and, from (78),

$$\operatorname{div}(f_0, g_0)(\psi_0^1(K,0)) = \operatorname{div}(f_0, g_0)(\psi_0^2(K,0)) = -K + O(K^2), \quad K \to 0$$

such that the function $\overline{V}_0^2(K)$ in formula (66) is given by:

$$\overline{V}_0^2(K) = -\frac{13}{24} + O(K), K \to 0.$$

Furthermore, one easily gets

$$\overline{V}_0^1(K) = O(K), K \to 0$$

implying that $\overline{V}_0(K)$ in formula (66) is given by:

$$\overline{V}_0(K) = \frac{13}{24} + O(K), K \to 0. \tag{80}$$

We are left with the calculations of the integrals appearing in the formulas (66) and (67) of $\eta_1(0)$ and $\eta_2(0)$ respectively.

Consider the integral in formula (67), along the orbit Γ_2 lying on the x–axis. Parametrizing the orbit using the x–coordinate leads to an integral over $x \in [r_1^1(0,K), r_1^2(0,K)]$ with $\psi_0^i = (r_1^i, r_2^i)$ and K varying in $]0, K_0[$, K_0 near zero. A direct calculation yields the following primitive of the integrand:

$$F(x) := \ln\left(\frac{1-x}{1+x}\right) - \frac{x}{x^2-1}.$$

Because $r_1^1(0,K) = -r_1^2(0,K)$ the integral equals:

$$\int_{\Gamma_K^2} \frac{\overline{\mathcal{A}}}{\parallel X_H \parallel^3}\, ds = -2\frac{r_1^2(0,K)}{r_1^2(0,K)^2 - 1} + 2\ln\left(\frac{1 - r_1^2(0,K)}{1 + r_1^2(0,K)}\right).$$

Using (73), one easily finds that

$$r_1^2(0,K) = 1 - K + \frac{1}{2}K^2 + O(K^3), \quad K \to 0$$

yielding in

$$\int_{\Gamma_K^2} \frac{\overline{\mathcal{A}}}{\| X_H \|^3} \, ds = \frac{1}{K} - 2\ln 2 + 2\ln K + O(K), \quad K \to 0. \tag{81}$$

Consider now the integral along the connection Γ_1 in formula (66),

$$\Gamma_1 : x^2 + \frac{1}{12} y^2 - 1 = 0,$$

which can be parametrized by the x–coordinate yielding an integral over $x \in [r_1^2(K,0), r_1^1(K,0)]$. A direct calculation shows that

$$G(x) := \frac{x\left(\sqrt{3}\pi\sqrt{3-3x^2}-4\right)}{8\left(1+11x^2\right)} - \frac{3}{4}\pi \arcsin x + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

is a primitive of the integrandum, where

$$g\left(x\right) = \frac{x\left(\sqrt{3}\pi\sqrt{3-3x^2}-4\right)}{8\left(1+11x^2\right)} - \frac{3}{4}\pi \arcsin x.$$

Thus, the integral equals $G(-r_1^2(K,0)) - G(r_1^2(K,0))$. Using (73), one sees

$$r_1^2(K,0) = 1 - \frac{K^2}{96} + O(K^3), \quad K \to 0$$

such that the integral equals

$$\int_{\Gamma_K^1} \frac{\overline{\mathcal{A}}}{\| X_H \|^3} \, ds = \frac{1}{12} + \frac{3}{4}\pi^2 + 2\ln K - \ln 192 + o(1), \quad K \to 0. \tag{82}$$

Substituting the obtained data (80), (82), (79) and (81) in the formulas (66) and (67), one gets:

$$\eta_1(0) = \frac{13}{12} + \ln 192 + \frac{3}{4}\pi^2 \quad \text{and} \quad \eta_2(0) = 2\ln 2.$$

Clearly, these values fulfilled the necessary condition $\eta_2\left(0\right) \neq 2\eta_1\left(0\right)$ in (11), we wanted to verify (see page 5).

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