

Cyclicity of unbounded semi-hyperbolic 2-saddle cycles in polynomial Liénard systems

Magdalena Caubergh¹, Freddy Dumortier² and Stijn Luca²

¹magdalena.caubergh@uab.es, Departament de Matemàtiques, Edifici C,

Universitat Autònoma de Barcelona, 08193 Cerdanyola de Vallès, Barcelona, Spain

²stijn.luca@uhasselt.be, freddy.dumortier@uhasselt.be, Departement Wiskunde,

Universiteit Hasselt, Campus Diepenbeek, Agoralaan-Gebouw D, B-3590 Diepenbeek, Belgium

Abstract

The paper deals with the cyclicity of unbounded semi-hyperbolic 2-saddle cycles in polynomial Liénard systems of type (m, n) with $m < 2n + 1$, m and n odd. We generalize the results in [1] (case $m = 1$), providing a substantially simpler and more transparent proof than the one used in [1].

1 Introduction

In this paper we will study families of Liénard systems

$$(X_{(a,b)}) : \begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^n a_i x^{n+1-i} \right), \\ \dot{y} &= - \left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i} \right), \end{cases} \quad (1)$$

with $(m, n) \in \mathbb{N}^2$, m and n odd and $m < 2n + 1$. We fix such (m, n) . As an important ingredient of the construction, we observe that $X_{(a,b)}$ is invariant under

$$(x, y, a^o, a^e, b^o, b^e, t) \mapsto (-x, y, -a^o, a^e, -b^o, b^e, -t), \quad (2)$$

with $a^o = (a_1, a_3, \dots, a_n)$, $a^e = (a_2, a_4, \dots, a_{n-1})$, $b^o = (b_1, b_3, \dots, b_{m-2})$ and $b^e = (b_2, b_4, \dots, b_{m-1})$.

The motivation to study systems (1) comes from the scalar equations:

$$\ddot{x} + Q(x)\dot{x} + P(x) = 0, \quad (3)$$

with P and Q polynomials of respective strict degrees m and n and with the highest degree coefficient of P positive. In the phase plane equation (3) can

be written as:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -P(x) - yQ(x). \end{cases} \quad (4)$$

By means of the transformation $\bar{y} = y + F(x)$, with $F'(x) = Q(x)$ and $F(0) = 0$, one can represent (4) in the so-called Liénard plane as:

$$\begin{cases} \dot{x} &= \bar{y} - F(x), \\ \dot{\bar{y}} &= -P(x). \end{cases} \quad (5)$$

After putting a singularity at the origin (which can always be done in case m and n are both odd) and a linear rescaling in (\bar{x}, \bar{y}, t) , expression (5) reduces to (1), at least for a good choice of the parameters (a, b) .

In aiming at studying the large amplitude limit cycles of (1) (see [1] for a definition), one uses a compactification of the plane described in [2] and [4]. The best way to do this is by using, near infinity:

$$(x, y) = \left(\frac{1}{r} \cos \theta, \frac{1}{r^{n+1}} \sin \theta \right), \quad (6)$$

for $r \sim 0$, and multiplying the obtained expression of (1) by r^n in order to desingularize. The procedure adds a circle at infinity (given by $\{r = 0\}$) near which the extended vector field (that we denote by $\bar{X}_{(a,b)}$) looks like in Figure 1. In this figure double arrows stand for hyperbolic behaviour and simple arrows for semi-hyperbolic behaviour.

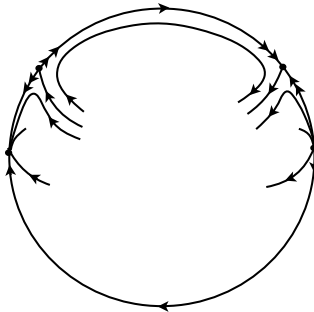


Figure 1: Behaviour near infinity of system (1).

Because of the chosen conditions on (m, n) , it is possible that for some parameter value

$$(\bar{a}, \bar{b}) = (\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_{m-1}),$$

$X_{(\bar{a}, \bar{b})}$ contains a heteroclinic connection between the two semi-hyperbolic saddles at infinity, giving rise to an *unbounded semi-hyperbolic 2-saddle cycle*.

In this paper we aim at finding an upperbound on the number of limit cycles that can perturb from such \mathcal{L} . When we say $(a, b) \sim (\bar{a}, \bar{b})$, we will restrict to

$$(a, b) = (a_1, \dots, a_n, b_1, \dots, b_{m-1}) \in \mathcal{W},$$

with \mathcal{W} a neighbourhood of (\bar{a}, \bar{b}) in parameter space.

This problem has been treated in [1] under the condition that $m = 1$. We want to generalize the results from [1] to the case $m > 1$. As shown in [4] unbounded semi-hyperbolic 2-saddle cycles only occur for Liénard systems of type (m, n) when $m < 2n + 1$ and both m and n are odd.

We are not yet able to provide a complete generalization for $m > 1$. We can however prove the results that follow below.

In Section 4, more precisely in (26), we define a sequence of polynomials $(P_i(a, b))_{i=1,2,\dots,N-1}$, with $N = 2n + 1 - m$, that reveal to play an important role in the subsequent calculations. More precisely we will consider the related sequence of polynomials

$$(c_j(a, b))_{j=1,\dots,K}, \tag{7}$$

for some $K \in \mathbb{N}$ that can be defined as follows:

$$\begin{aligned} c_1 &= P_{2i_1+1}, \\ &\quad \text{if } P_1 \equiv P_3 \equiv P_5 \equiv \dots \equiv P_{2i_1-1} \equiv 0 \text{ and } P_{2i_1+1} \neq 0, \\ c_2 &= P_{2i_2+1} \pmod{c_1}, \\ &\quad \text{if } P_1, P_3, \dots, P_{2i_2-1} \text{ belong to the ideal generated by } c_1 \text{ and } P_{2i_2+1} \\ &\quad \text{does not,} \\ &\vdots \\ c_l &= P_{2i_l+1} \pmod{(c_1, \dots, c_{l-1})}, \\ &\quad \text{if } P_1, P_3, \dots, P_{2i_l-1} \text{ belong to the ideal generated by } (c_1, \dots, c_{l-1}) \\ &\quad \text{and } P_{2i_l+1} \text{ does not.} \end{aligned}$$

We end with $l = K$ in a way that all $(P_1, P_3, \dots, P_{N-1})$ belong to the ideal generated by (c_1, \dots, c_K) . We see that $K \leq N/2$.

We call (c_1, \dots, c_K) the leading *large amplitude Lyapunov quantities* of the chosen family.

Remarks:

1. In the process just described we can choose for c_l the polynomial P_{2i_l+1} itself or any other (preferably simpler) polynomial that is equal to $P_{2i_l+1} \pmod{(c_1, \dots, c_{l-1})}$.
2. In the process just described there is no need to work with a full family of Liénard systems as in (1), we can also work with a subfamily. In

case of a full family it will reveal that $K = N/2$, while for a subfamily it can be strictly smaller. We will only work with subfamilies that are obtained from (1) by restricting one or more parameters a_i or b_j to a constant value, while keeping the other; let us call them *full subfamilies*.

We can now formulate our main theorem.

Theorem 1 *Let $(Y_{(a,b)})$ be a full subfamily of (1) and let $c_1(a,b), \dots, c_K(a,b)$ be the related leading large amplitude Lyapunov quantities. Let $(a,b) = (\bar{a}, \bar{b})$ be a value for which $Y_{(\bar{a}, \bar{b})}$ is defined and has an unbounded semi-hyperbolic 2-saddle cycle $\mathcal{L}_{(\bar{a}, \bar{b})}$. Then*

- (i) *If $c_k(\bar{a}, \bar{b}) \neq 0$ for some $1 \leq k \leq K$, then the cyclicity of $\mathcal{L}_{(\bar{a}, \bar{b})}$ in the family $Y_{(\bar{a}, \bar{b})}$ is bounded by k .*
- (ii) *If $c_1(\bar{a}, \bar{b}) = \dots = c_K(\bar{a}, \bar{b}) = 0$, and if $Y_{(\bar{a}, \bar{b})}$ has a center near infinity and if $(a,b) \mapsto (c_1(a,b), \dots, c_K(a,b))$ is a submersion at (\bar{a}, \bar{b}) , then the cyclicity of $\mathcal{L}_{(\bar{a}, \bar{b})}$ in the family $(Y_{(a,b)})$ is bounded by K .*

Remark: Instead of saying *cyclicity of $\mathcal{L}_{(\bar{a}, \bar{b})}$* in the statement of Theorem 1, we can also say *large amplitude limit cycles of $Y_{(a,b)}$ for $(a,b) \sim (\bar{a}, \bar{b})$* .

Theorem 1 does not permit to treat the cyclicity problem of unbounded semi-hyperbolic 2-saddle cycles in families (1) completely. It however induces a complete answer for a number of interesting cases, including the classical Liénard equations that have been treated in [1].

The main problem in trying to apply Theorem 1, consists in calculating the $\{c_j(a,b) \mid 1 \leq j \leq K\}$. This is rather easy in case for each i we have either $a_{2i+1} = 0$ or $b_{2i+1} = 0$. In the other case the calculation might get quite involved. This is similar to the kind of problems that are encountered in calculating Lyapunov quantities at a non-degenerate singularity of center-focus type.

In Section 9 we will show that following theorems can be obtained as corollaries of Theorem 1.

Theorem 2 *Consider a full subfamily $(Y_{(a,b)})$ of (1) such that (where we write $a_i = 0$ when $i > n$ and $b_i = 0$ when $i \geq m$):*

- (i) *$a_{2i+1}b_{2i+1} = 0$, when $1 \leq 2i+1 \leq 2n-m$,*
- (ii) *$a_{2i+1} = b_{2i+1} = 0$, when $2i+1 > 2n-m$.*

Let $(a,b) = (\bar{a}, \bar{b})$ be a value, satisfying the above conditions, such that $Y_{(\bar{a}, \bar{b})}$ has an unbounded semi-hyperbolic 2-saddle cycle $\mathcal{L}_{(\bar{a}, \bar{b})}$. Then:

$$\text{Cycl}((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq \#\{j \mid 1 \leq 2j+1 \leq 2n-m, a_{2j+1}^2 + b_{2j+1}^2 \neq 0\}.$$

Special cases are the following of which Theorem 4 contains the theorem proven in [1].

Theorem 3 *Let $(Y_{(a,b)})$ be a family of polynomial Liénard equations of type (m, n) , $m \leq n$, with odd friction term:*

$$(Y_{(a,b)}) : \begin{cases} \dot{x} = y - \left(x^{n+1} + \sum_{i=1}^{(n-1)/2} a_{2i} x^{n+1-2i} \right), \\ \dot{y} = - \left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i} \right), \end{cases}$$

such that $Y_{(a,b)}$ has an unbounded semi-hyperbolic 2-saddle cycle at $(a, b) = (\bar{a}, \bar{b})$. Then $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq \#\{j \mid b_{2j+1} \neq 0\} \leq \frac{m-1}{2}$.

Theorem 4 *Let $(Y_{(a,b)})$ be a family of polynomial Liénard equations of type (m, n) , $m \leq n$, with odd forcing term.*

$$(Y_{(a,b)}) : \begin{cases} \dot{x} = y - \left(x^{n+1} + \sum_{i=1}^n a_i x^{n+1-i} \right), \\ \dot{y} = - \left(x^m + \sum_{i=1}^{(m-1)/2} b_{2i} x^{m-2i} \right), \end{cases}$$

such that $Y_{(a,b)}$ has an unbounded semi-hyperbolic 2-saddle cycle at $(a, b) = (\bar{a}, \bar{b})$. Then $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq \#\{j \mid a_{2j+1} \neq 0\} \leq \frac{n+1}{2}$.

Remark: When $m > n$ ($\Rightarrow 2n - m < n < m$) in the 2 previous theorems, condition (ii) in Theorem 2 imposes extra conditions on the families of Liénard equations with odd forcing or odd friction term.

Of course it might be possible to apply Theorem 1 to specific cases in which some $a_{2i+1}b_{2i+1}$ are not identically zero. As an example we prove:

Theorem 5 *Consider the following full subfamily of (1):*

$$(Y_{(a,b)}) : \begin{cases} \dot{x} = y - (x^6 + a_1 x^5 - x^4 + a_4 x^2), \\ \dot{y} = -(x^3 + b_1 x^2 + x). \end{cases}$$

Let $(a, b) = (\bar{a}_1, \bar{b}_1, \bar{a}_4)$ be such that $Y_{(a,b)}$ has an unbounded semi-hyperbolic 2-saddle cycle $\mathcal{L}_{(\bar{a}, \bar{b})}$, then $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq 2$

The methods that we will use in the proof of these results are partly inspired from [1]. However, based on a recent normal linearization theorem at semi-hyperbolic singularities (see [3]) and a systematic use of the symmetry

(2) we obtain a substantial simplification of the calculations made in [1], providing a more transparent proof.

We will now first introduce, as in [1], a *difference map* near the unbounded semi-hyperbolic 2-saddle cycle in a way that large amplitude limit cycles agree with small positive zeros of the difference map.

2 Study near infinity

As is usual, in working with a compactification as given in (6), it is preferable to work with charts. For a system $X_{(a,b)}$ like in (1), both semi-hyperbolic saddles at infinity lie in the chart in the positive y -direction. This is obtained by means of the transformation

$$x = u/s, \quad y = 1/s^{n+1}$$

and by multiplying the result with s^n , leading to the family

$$(\hat{X}_{(a,b)}) : \begin{cases} \dot{u} = 1 - u^{n+1} - \sum_{i=1}^n a_i u^{n+1-i} s^i \\ \quad + \frac{u^{m+1}}{n+1} s^{2n+1-m} + \frac{1}{n+1} u \sum_{i=1}^{m-1} b_i u^{m-i} s^{2n+1-m+i}, \\ \dot{s} = \frac{1}{n+1} s^{2n+2-m} \left(u^m + \sum_{i=1}^{m-1} b_i u^{m-i} s^i \right). \end{cases} \quad (8)$$

Because of the invariance of $X_{(a,b)}$ under transformation (2), we see that $\hat{X}_{(a,b)}$ is invariant under

$$(u, s, a^o, a^e, b^o, b^e, t) \mapsto (-u, s, -a^o, a^e, -b^o, b^e, -t). \quad (9)$$

System $(\hat{X}_{(a,b)})$ has two singularities $s_{\pm} = (\pm 1, 0)$ that are both semi-hyperbolic saddles with linear part

$$\pm \begin{pmatrix} -(n+1) & a_1 \\ 0 & 0 \end{pmatrix}.$$

The behaviour near s_- follows easily from the behaviour near s_+ using that $\hat{X}_{(a,b)}$ is invariant under the transformation

$$(t, u, s) \mapsto (-t, -u, -s). \quad (10)$$

Therefore the behaviour of the flow of $-\hat{X}_{(a,b)}$ near s_- in the region $\{(u, s) \mid s > 0, -1 < u < -\varepsilon\}$ is found precisely in the behaviour of the flow of $\hat{X}_{(a,b)}$ near s_+ in the region $\{(u, s) \mid s < 0, \varepsilon < u < 1\}$, where $\varepsilon > 0$.

Combining (9) and (10), we see that $\hat{X}_{(a,b)}$ is also invariant under the composition:

$$(u, s, a^o, a^e, b^o, b^e, t) \mapsto (u, -s, -a^o, a^e, -b^o, b^e, t). \quad (11)$$

To understand the behaviour near $s_+ = (1, 0)$, we study the behaviour on a center manifold $W_{(a,b)}^c$ that locally can be written as a graphic

$$\{(1 + u_0(s, a, b), s)\}, \quad (12)$$

for some smooth function u_0 with $u_0(0, a, b) = 0$. From (8) it is clear that we can write

$$u_0(s, a, b) = \varphi(s) + \psi(s, a, b), \quad (13)$$

with $\varphi(s) = \frac{1}{(n+1)^2} s^{2n+1-m} (1 + O(s))$ and $\psi(s, a, b) = O(a, b)$, $(a, b) \rightarrow (0, 0)$; moreover φ is an even function in s .

Since $\hat{X}_{(a,b)}$ is invariant under (11), we also know that the (a, b) -family of center manifolds is invariant under the same transformation. We hence obtain that $u_0 = u_0(s, a, b)$ is invariant under

$$(s, a^o, a^e, b^o, b^e) \mapsto (-s, -a^o, a^e, -b^o, b^e). \quad (14)$$

The behaviour on a center manifold is given by

$$\dot{s} = \frac{1}{n+1} s^{2n+2-m} + O(s^{2n+3-m}), \quad s \rightarrow 0. \quad (15)$$

Since m is odd, $\hat{X}_{(a,b)}$ is of saddle-type near s_+ and the center manifold is unique.

Proposition 6 *Let $\{u = 1 + u_0(s, a, b)\}$ be the center manifold of $\hat{X}_{(a,b)}$ at s_+ , then u_0 can be written as*

$$u_0(s, a, b) = \sum_{i=1}^{2n-m} \nu_i(a) s^i + O(s^{2n+1-m}), \quad s \rightarrow 0, \quad (16)$$

where

$$(i) \quad \nu_1(a) = -\frac{1}{n+1} a_1, \quad \text{and}$$

$$(ii) \quad \nu_i(a) = -\frac{1}{n+1} a_i + O(a_1, \dots, a_{i-1}),$$

for all $1 \leq i \leq 2n - m$, with $a_i = 0$ if $i > n$. Moreover

$$(iii) \quad \nu_{2j+1}(a) = -\frac{1}{n+1} a_{2j+1} + O(a_1, a_3, \dots, a_{2j-1}),$$

for all j with $1 \leq j \leq \frac{1}{2}(2n - m - 1)$, and where $a_{2j+1} = 0$ if $2j + 1 > n$.

Proof. Using the invariance of $u = u_0(s) = u_0(s, a, b)$ under the flow of $\hat{X}_{(a,b)}$, one finds:

$$\begin{aligned} \frac{du_0}{ds}(s) \frac{1}{n+1} s^{2n+2-m} \left((1+u_0(s))^m + \sum_{i=1}^{m-1} b_i (1+u_0(s))^{m-i} s^i \right) \\ = \\ 1 - (1+u_0(s))^{n+1} - \sum_{i=1}^n a_i (1+u_0(s))^{n+1-i} s^i + \frac{1}{n+1} (1+u_0(s))^{m+1} s^{2n+1-m} \\ + \frac{1}{n+1} (1+u_0(s)) \sum_{i=1}^{m-1} b_i (1+u_0(s))^{m-i} s^{2n+1-m+i}. \end{aligned}$$

Up to order $O(s^{2n+1-m})$, $s \rightarrow 0$ the above equation is given by:

$$1 - (1+u_0(s))^{n+1} - \sum_{i=1}^n a_i (1+u_0(s))^{n+1-i} s^i = 0. \quad (17)$$

This already shows that the ν_i do not depend on b .

We continue by substituting

$$u_0(s, a, b) = \nu_1 s + \nu_2 s^2 + \dots + \nu_{2n-m} s^{2n-m} + O(s^{2n+1-m}), s \rightarrow 0,$$

in (17). The result will follow by induction on i .

Comparing coefficients with s in (17), we find $\nu_1 = -\frac{a_1}{n+1}$. The second statement easily follows by considering (17) under the extra condition that $a_1 = \dots = a_{i-1} = 0$. For the third statement, we take k such that $a_1 = a_3 = \dots = a_{2k-1} = 0$. By induction $\nu_1 = \nu_3 = \dots = \nu_{2k-1} = 0$, reducing equation (17) to

$$\begin{aligned} 1 - \left(1 + \sum_{j=1}^k \nu_{2j} s^{2j} + \nu_{2k+1} s^{2k+1} \right)^{n+1} - \sum_{j=1}^k a_{2j} (1+u_0(s))^{n+1-2j} s^{2j} \\ - a_{2k+1} (1+u_0(s))^{n+1-2k-1} s^{2k+1} = O(s^{2k+2}), s \rightarrow 0. \end{aligned}$$

Comparing coefficients with s^{2k+1} this leads to $\nu_{2k+1} = -\frac{a_{2k+1}}{n+1}$. \blacksquare

Remark: Since u_0 is invariant under (14) we also see that

$$\nu_{2j+1}(-a^0, a^e) = -\nu_{2j+1}(a^0, a^e) \quad \text{and} \quad \nu_{2j}(-a^0, a^e) = \nu_{2j}(a^0, a^e). \quad (18)$$

3 The difference map

We will now introduce, in a different way than in [1], an appropriate difference map near an unbounded semi-hyperbolic 2-saddle cycle \mathcal{L} as represented in Figure 2. We strongly refer to that figure for the notions and notations that we will introduce now.

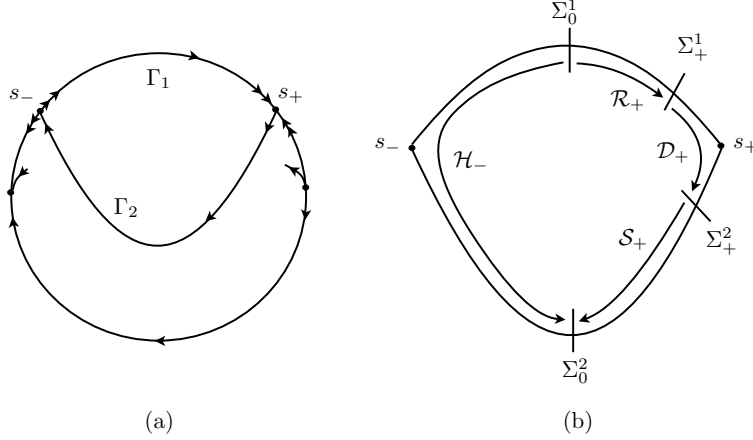


Figure 2: (a) An unbounded semi-hyperbolic 2-saddle cycle \mathcal{L} . (b) Regular transition maps and Dulac maps near \mathcal{L} .

Denote by Γ_1 the connection at infinity between the saddles s_- and s_+ of $\bar{X}_{(a,b)}$, i.e. the part of the u -axis lying between $(-1, 0)$ and $(1, 0)$ for $\hat{X}_{(a,b)}$. This connection stays fixed for all parameter values $(a, b) \sim (\bar{a}, \bar{b})$. Denote by $\Gamma_{2,a,b}^-$ and $\Gamma_{2,a,b}^+$ the respective center manifolds of s_- and s_+ . For $(a, b) = (\bar{a}, \bar{b})$, these two manifolds coincide along a heteroclinic connection, being part of \mathcal{L} . Denote this connection by Γ_2 .

Choose sections Σ_+^i and Σ_0^i transverse to Γ_i and parametrized by a regular parameter. Σ_+^i are chosen near the saddle s_+ , while Σ_0^i are chosen on $\{x = 0\}$. Furthermore, if the regular parameter on Σ_0^1 is denoted by w , $w > 0$, then we suppose that the intersection $\Gamma_1 \cap \Sigma_0^1$ corresponds to $w = 0$.

One defines:

1. the regular transition map $\mathcal{R}_+^{(a,b)}$ near Γ_1 from Σ_0^1 to Σ_+^1 , defined by the flow of $\bar{X}_{(a,b)}$,
2. the Dulac map $\mathcal{D}_+^{(a,b)}$ describing the corner passage near s_+ from Σ_+^1 to Σ_+^2 defined by the flow of $\bar{X}_{(a,b)}$,
3. the regular transition maps $\mathcal{S}_+^{(a,b)}$ near Γ_2 from Σ_+^2 to Σ_0^2 , defined by the flow of $\bar{X}_{(a,b)}$,
4. the transition maps $\mathcal{H}_\pm^{(a,b)} = \mathcal{H}_\pm(w, a, b)$ near \mathcal{L} from Σ_0^1 to Σ_0^2 , defined by the flow of $\pm \bar{X}_{(a,b)}$. In particular $\mathcal{H}_+^{(a,b)} = \mathcal{S}_+^{(a,b)} \circ \mathcal{D}_+^{(a,b)} \circ \mathcal{R}_+^{(a,b)}$.

By the invariance of $X_{(a,b)}$ under (2) one has:

$$\mathcal{H}_-(w, a^o, a^e, b^o, b^e) = \mathcal{H}_+(w, -a^o, a^e, -b^o, b^e).$$

The difference map $\Delta^{(a,b)} : \Sigma_0^1 \mapsto \Sigma_0^2$, expressed in the chosen parameters on the sections Σ_0^1 and Σ_0^2 , can now be defined as:

$$\begin{aligned} \Delta^{(a,b)}(w) &= \Delta(w, a, b) = \\ \mathcal{H}_+(w, -a^o, a^e, -b^o, b^e) &- \mathcal{H}_+(w, a^o, a^e, b^o, b^e), \end{aligned} \quad (19)$$

for $w \sim 0$ and $(a, b) \sim (\bar{a}, \bar{b})$. Notice that $\Delta(w, 0, a^e, 0, b^e) = 0$.

The large amplitude limit cycles correspond to small positive zeros of $\Delta^{(a,b)}$. The cyclicity $Cycl(\bar{X}_{(a,b)}, \mathcal{L}, (\bar{a}, \bar{b}))$ is equal to the least upper bound of the number of isolated zeros of $\Delta^{(a,b)}$, for $w \downarrow 0, (a, b) \rightarrow (\bar{a}, \bar{b})$. An upperbound on this cyclicity will be found by applying a division-derivation algorithm to $\Delta^{(a,b)}$, based on Rolle's theorem.

4 Normalizing coordinates

Choosing appropriate normalizing coordinates near the semi-hyperbolic saddles at infinity will appear to be a helpful tool in simplifying the calculation of the difference map.

We can and will restrict to considering s_+ and we change $\hat{X}_{(a,b)}$ near s_+ by the equivalent family $Y_{(a,b)}$ defined as:

$$Y_{(a,b)} = (n+1) \left(u^m + \sum_{i=1}^{m-1} b_i u^{m-i} s^i \right)^{-1} \hat{X}_{(a,b)}. \quad (20)$$

We now introduce $z = u - (1 + u_0)$ with $u_0 = u_0(s, a, b)$ and write

$$u^m + \sum_{i=1}^{m-1} b_i u^{m-i} s^i = \alpha_1(s, a, b) + \alpha_2(s, a, b)z + O(z^2),$$

with

$$\alpha_1(s, a, b) = (1 + u_0)^m + \sum_{i=1}^{m-1} b_i (1 + u_0)^{m-i} s^i,$$

and

$$\alpha_2(s, a, b) = m(1 + u_0)^{m-1} + \sum_{i=1}^{m-1} b_i (m-i)(1 + u_0)^{m-i-1} s^i.$$

We also write $\dot{u} = \beta_1(s, a, b) + \beta_2(s, a, b)z + O(z^2)$, with

$$\begin{aligned} \beta_1(s, a, b) &= 1 - (1 + u_0)^{n+1} - \sum_{i=1}^n a_i (1 + u_0)^{n+1-i} s^i \\ &+ \frac{(1 + u_0)^{m+1}}{n+1} s^{2n+1-m} + \frac{1}{n+1} (1 + u_0) \sum_{i=1}^{m-1} b_i (1 + u_0)^{m-i} s^{2n+1-m+i}, \end{aligned}$$

and

$$\begin{aligned} \beta_2(s, a, b) = & -(n+1)(1+u_0)^n + \frac{m+1}{n+1}(1+u_0)^m s^{2n+1-m} \\ & - \sum_{i=1}^n a_i(n+1-i)(1+u_0)^{n-i} s^i \\ & + \frac{1}{n+1} \left(\sum_{i=1}^{m-1} b_i(m+1-i)(1+u_0)^{m-i} s^{2n+1-m+i} \right). \end{aligned}$$

In the new coordinates (z, s) , the family $Y_{(a,b)}$, as defined in (20), can be written as:

$$\begin{cases} \dot{z} &= -A(s, a, b)z + O(z^2), \\ \dot{s} &= s^{2n+2-m}, \end{cases} \quad (21)$$

with

$$A(s, a, b) = \frac{n+1}{\alpha_1(s, a, b)^2} (\alpha_2(s, a, b)\beta_1(s, a, b) - \alpha_1(s, a, b)\beta_2(s, a, b)). \quad (22)$$

The function $A(s, a, b)$ is strictly positive and is invariant under

$$(s, a^o, a^e, b^o, b^e) \mapsto (-s, -a^o, a^e, -b^o, b^e). \quad (23)$$

From Theorem 1.3 of [3], we know that, on a (a, b) -uniform neighbourhood V of (\bar{a}, \bar{b}) , there exists a smooth (a, b) -family of coordinate changes of the form

$$(Z, s) = (z(1 + z\bar{Z}(z, s, a, b)), s), \quad (24)$$

conjugating (21) to

$$\begin{cases} \dot{Z} &= -A(s, a, b)Z, \\ \dot{s} &= s^{2n+2-m}, \end{cases} \quad (25)$$

for the same $A(s, a, b)$ as in (21). The same can be done for $(a, b) \sim (-\bar{a}^o, \bar{a}^e, -\bar{b}^o, \bar{b}^e)$ and the coordinate change can even be chosen to commute with (23) in the sense that also $\bar{Z}(z, s, a, b)$ is invariant under (23); we will however not have to rely on the latter.

From (22), we know that A is given by:

$$A(s, a, b) = (n+1)^2 + \sum_{i=1}^{N-1} P_i(a, b)s^i + s^N(n-2m-1 + P_N(a, b)) + O(s^{N+1}), \quad (26)$$

as $s \rightarrow 0$ for some polynomials $P_i(a, b)$, $1 \leq \forall i \leq N$, with $P_i(0, 0) = 0$ and where $N = 2n + 1 - m$.

In order to calculate the difference map, we will now first study the expression of the coefficients $P_i(a, b)$.

5 Expression of the coefficients $P_i(a, b)$.

As in [1] it will reveal that the most interesting information will be contained in the functions $P_{2k+1}(a, b)$, with $0 \leq 2k + 1 \leq N - 1$, and more specifically in the related functions $c_1(a, b), \dots, c_K(a, b)$ as defined in (7). Each $P_{2k+1}(a, b)$ is an algebraic expression in (a, b) of which the complexity increases considerably as k is getting bigger.

In the case $m = n = 3$, we have for instance:

$$\begin{aligned} P_1(a, b) &= 16\left(\frac{3}{4}a_1 - b_1\right), \\ P_3(a, b) &= 2a_1(2b_2 - a_2) + 4a_3 + O\left(\frac{3}{4}a_1 - b_1\right). \end{aligned}$$

When $m = 3, n = 5$, one has:

$$\begin{aligned} P_1(a, b) &= 36\left(\frac{1}{2}a_1 - b_1\right), \\ P_3(a, b) &= 2a_1(3b_2 - 2a_2) + \frac{7}{9}a_1^3 + 6a_3 + O\left(\frac{1}{2}a_1 - b_1\right), \end{aligned} \tag{27}$$

and

$$P_5(a, b) = 2a_1a_4 - 4a_1a_2b_2 + \frac{4}{3}a_1^3b_2 - \frac{1}{54}a_1^5 - 6a_5 + 6a_1b_2^2,$$

given that $P_1(a, b) = P_3(a, b) = 0$.

It appears that the complexity of the expression of $P_{2k+1}(a, b)$ is caused by the mixing of terms $\{(a_{2i+1}, b_{2i+1}) \mid 0 \leq i \leq k\}$, that appear in (1). The following theorem can however easily be proven.

Theorem 7 *Let $A(s, a, b)$ be the C^∞ function defined in (22) and let P_i be the coefficients defined in (26). Then*

$$(i) \quad P_i(-a^o, a^e, -b^o, b^e) = (-1)^i P_i(a^o, a^e, b^o, b^e),$$

for $1 \leq i \leq 2n - m + 1$. Moreover (and here we write $a_i = 0$ for $i > n$ and $b_i = 0$ for $i \geq m$)

$$(ii) \quad P_i(a, b) \text{ only depends on } (a_1, \dots, a_i, b_1, \dots, b_i),$$

and when $(a_1, a_3, \dots, a_{2k-1}, b_1, b_3, \dots, b_{2k-1}) = 0$, then

$$(iii) \quad P_{2j+1}(a, b) = 0, \quad \forall 0 \leq j \leq k - 1,$$

and

$$(iv) \quad P_{2k+1}(a, b) = (n + 1)^2 \left(\frac{m-2k}{n+1} a_{2k+1} - b_{2k+1} \right).$$

Proof. (i) follows immediately from the fact that $A(s, a, b)$ is invariant under (14).

(ii) follows from the fact that in the expression of $A(s, a, b)$ a_i and b_i are always accompanied by s^i (or s^j , with $j > i$).

(iii) is a direct consequence of (i) and (ii).

(iv) is easily obtained by putting $(a_1, a_3, \dots, a_{2k-1}, b_1, b_3, \dots, b_{2k-1}) = 0$ in the expression of $A(s, a, b)$ and checking the coefficient in front of s^{2k+1} .

■

6 Dulac map near s_+

In this section we express the Dulac map D_+ in normalizing coordinates, i.e. D_+ calculated from $\{Z = 1\}$ to $\{s = s_0\}$, with $s_0 > 0$, for the expression (25). Let us call it *the normalized expression of $\mathcal{D}_+^{(a,b)}$* .

Proposition 8 *Let $D_+(s, a, b)$ be the normalized expression of the Dulac map $\mathcal{D}_+^{(a,b)}$ near the saddle s_+ and defined for $s \geq 0$. Then, denoting $N = 2n + 1 - m$:*

$$(i) \quad D_+(s, a, b) = \exp\left(-\frac{1}{s^N} (F(s) + G_+(s, a, b))\right),$$

for some functions $F(s)$ and $G_+(s, a, b)$ with $F(0) = \frac{(n+1)^2}{N}$. Moreover

$$(ii) \quad F(s) = \frac{(n+1)^2}{N} - (n - 2m - 1)s^N \ln s + O(s^N), \quad s \rightarrow 0,$$

and

$$(iii) \quad G_+(s, a, b) = \sum_{i=1}^{N-1} \frac{P_i(a,b)}{N-i} s^i - P_N(a,b) s^N \ln s + s^N \Phi(s, a, b), \quad s \rightarrow 0,$$

for some C^∞ function $\Phi(s, a, b)$.

Proof. Clearly

$$D_+(s) = \exp\left(\int_{s_0}^s \frac{A(x, a, b)}{x^{2n+2-m}} dx\right). \quad (28)$$

Now, because of (26):

$$\begin{aligned} & \int_{s_0}^s \frac{A(x, a, b)}{x^{2n+2-m}} dx \\ &= \int_{s_0}^s \frac{1}{x^{N+1}} \left((n+1)^2 + \sum_{i=1}^{N-1} P_i(a, b) x^i \right. \\ & \quad \left. + x^N ((n - 2m - 1) + P_N(a, b)) + O(x^{N+1}) \right) dx, \end{aligned} \quad (29)$$

$$\begin{aligned}
 &= \int_{s_0}^s \left(\frac{(n+1)^2}{x^{N+1}} + \sum_{i=1}^{N-1} P_i(a,b) \frac{1}{x^{N+1-i}} \right. \\
 &\quad \left. + \frac{1}{x} ((n-2m-1) + P_N(a,b)) + O(1) \right) dx, \\
 &= \left[-\frac{(n+1)^2}{N} \frac{1}{x^N} - \sum_{i=1}^{N-1} \frac{P_i(a,b)}{N-i} \frac{1}{x^{N-i}} \right. \\
 &\quad \left. + ((n-2m-1) + P_N(a,b)) \ln x \right]_{s_0}^s + O(1) \\
 &= -\frac{(n+1)^2}{N} \frac{1}{s^N} - \sum_{i=1}^{N-1} \frac{P_i(a,b)}{N-i} \frac{1}{s^{N-i}} \\
 &\quad + ((n-2m-1) + P_N(a,b)) \ln s + O(1),
 \end{aligned} \tag{30}$$

as $s \rightarrow 0$. The result now easily follows. ■

7 Further study of the difference map

Denote by $\varphi(u, s) = \varphi_{(a,b)}(u, s) = (\psi_{(a,b)}(u, s), s)$ the (a, b) -family of coordinate changes conjugating $(Y_{(a,b)})$, as defined in (20), to (25) locally near s_+ . φ is the succession of the mapping introducing z , defined below (20), and the mapping defined in (24).

Let

$$\sigma_+^1 = \{(1, s) \mid s \geq 0\} \text{ and } \sigma_+^2 = \{(Z, s_0) \mid Z \geq 0\}.$$

By taking $s_0 > 0$ sufficiently small and after some dilatation in Z one can suppose that these sections lie in the domain of φ^{-1} .

We now choose Σ_+^2 in the (u, s) -coordinates near s_+ inside $\{s = s_0\}$ and transversally cutting the center manifold $\Gamma_{2,a,b}^+$. We choose $\Sigma_+^1 \subset \varphi^{-1}(\{Z = 1\})$ transversally cutting Γ_1 (see Figure 2 (b) and 3). We parametrize Σ_+^2 by $\{Z \geq 0\}$, through the mapping φ , ($Z=0$ represents $\Gamma_{2,a,b}^+$). We parametrize Σ_+^1 , also through φ , by $s \geq 0$. All sections depend, in a smooth way, on (a, b) . The section Σ_0^1 (resp. Σ_0^2) is chosen inside $\{u = 0\}$ (resp. $\{x = 0\}$) transverse to Γ_1 (resp. Γ_2) and parametrized by $s > 0$ (resp. y).

As in Section 6, the Dulac map $\mathcal{D}_+^{(a,b)}$ can be studied in the normalizing coordinates in the quadrant $\{Z > 0, s > 0\}$, (from σ_+^1 to σ_+^2). Parametrizing σ_+^1 by s and σ_+^2 by $Z > 0$, we get the expressions given in Proposition 8 for

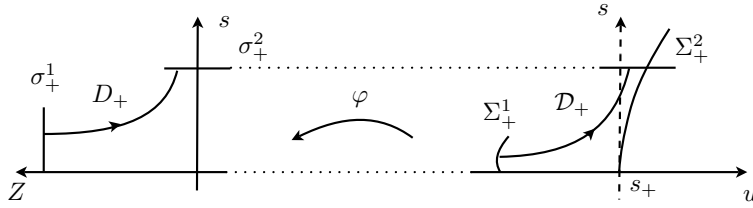


Figure 3: Corner passages expressed using normalizing coordinates.

$\mathcal{D}_+^{(a,b)}$, namely: $\mathcal{D}_+^{(a,b)}$ is expressed by:

$$(1, s) \mapsto (D_+(s, a, b), s_0).$$

In a similar way the regular transition maps $\mathcal{R}_+^{(a,b)}$, $\mathcal{S}_+^{(a,b)}$ can be represented as going from Σ_0^1 to σ_+^1 and from σ_+^2 to Σ_0^2 respectively. One obtains maps $R_+(s, a, b)$, $S_+(y, a, b)$ respectively. R_+ is expressed in $s > 0$ with values in $s > 0$, S_+ is expressed in $Z > 0$ with values in y . Furthermore let us denote

$$H_+(s, a, b) = S_+(D_+(R_+(s, a, b), a, b), a, b), \quad (31)$$

going from Σ_0^1 to Σ_0^2 .

The difference map as defined in (19) can now be expressed as:

$$\Delta(s, a^o, a^e, b^o, b^e) = H_+(s, -a^o, a^e, -b^o, b^e) - H_+(s, a^o, a^e, b^o, b^e), \quad (32)$$

for small positive values of s .

Let us now first provide nice expressions for the regular transition map R_+ using *normalized* sections and parametrizations.

Choose $u_0 \in]0, 1[$ in (u, s) -coordinates such that $\psi_{(a,b)}(u_0, 0) = 1$. Consider the sections $\{u = u_0\}$ and $\{u = 0\}$ transversally cutting the u -axis and parametrized by the s -coordinate. Let $T_+(s, a, b)$ be the regular transition map from $\{u = 0\}$ to $\{u = u_0\}$. In the following lemma we give an expression for this regular transition maps T_+ .

Lemma 9 Consider the family $(\hat{X}_{(a,b)})$ (8). Consider the regular transition maps T_+ from $\{u = 0\}$ to $\{u = u_0\}$. Then:

$$T_+(s, a, b) = s + O(s^{2n+2-m}), \quad s \rightarrow 0,$$

Proof. We transform (8) into the equivalent differential equation

$$\frac{ds}{du} = s^{2n+2-m} Q(u, s, a, b), \quad (33)$$

with Q the rational function given by:

$$Q(u, s, a, b) = \frac{1}{n+1} \frac{u^m + \sum_{i=1}^{m-1} b_i u^{m-i} s^i}{(1-u^{n+1}) \left(1 - \frac{P(u,s,a,b)}{1-u^{n+1}}\right)},$$

and where $P(u, s, a, b)$ is the polynomial:

$$P(u, s, a, b) = \sum_{i=1}^n a_i u^{n+1-i} s^i - \frac{u^{m+1}}{n+1} s^{2n+1-m} - \frac{1}{n+1} u \sum_{i=1}^{m-1} b_i u^{m-i} s^{2n+1-m+i}.$$

Clearly, $Q(u, s, a, b)$ is bounded on every compact subset of $] - 1, 1[\times \mathbb{R} \times \mathbb{R}^{n+m-1}$.

The solution of equation (33) is $s(0, s_0, u, a, b)$ with initial conditions $s(0, s_0, 0, a, b) = s_0$. In particular, we have

$$\frac{\partial}{\partial u} s(0, s_0, u, a, b) = (s(0, s_0, u, a, b))^{2n+2-m} Q(u, s(0, s_0, u, a, b), a, b). \quad (34)$$

and $T_+(s_0, a, b) = s(0, s_0, u_0, a, b)$. Clearly, for $s_0 \sim 0$ and $(a, b) \sim (\bar{a}, \bar{b})$:

$$\{(u, s(0, s_0, u, a, b)) \mid u \in [-u_0, u_0]\} \subset K,$$

for some compact $K \subset] - 1, 1[\times \mathbb{R} \times \mathbb{R}^{n+m-1}$. Therefore integrating both sides of equation (34) from $u = 0$ to $u = u_0$:

$$\begin{aligned} |T_+(s_0) - s_0| &= \left| \int_0^{u_0} (s(0, s_0, u, a, b))^{2n+2-m} Q(u, s(0, s_0, u, a, b), a, b) du \right| \\ &\leq M \int_0^{u_0} |s(0, s_0, u, a, b)|^{2n+2-m} du, \end{aligned}$$

for some $M > 0$. Because $s(0, s_0, u, a, b) = O(s_0)$, $s_0 \rightarrow 0$, uniformly for $u \in [-u_0, u_0]$, (i) immediately follows. ■

Choose sections $\pi_{(a,b)}^+ \subset \varphi(\{u = u_0\})$. We parametrize $\pi_{(a,b)}^+$ through φ by $s \geq 0$. Consider the regular transition map $F_+ : \pi_{(a,b)}^+ \mapsto \sigma_+^1$ expressed using the chosen parametrisation on $\pi_{(a,b)}^+$, σ_+^1 . Then we have

$$R_+(s, a, b) = F_+(T_+(s, a, b), a, b).$$

Lemma 10 *Let $F_+(s, a, b)$ be the regular transition map from $\pi_{(a,b)}^+$ to σ_+^1 . Then*

$$F_+(s, a, b) = s + O(s^{2n+2-m}), s \rightarrow 0.$$

Proof. To shorten notation we write $F_+(s) = F_+(s, a, b)$. The map $F_+(s)$ is defined by the integral equation:

$$\int_{\psi_{(a,b)}(u_0, s)}^1 \frac{1}{x} dx = \int_s^{F_+(s)} \frac{A(x, a, b)}{x^{2n+2-m}} dx.$$

Totally similar as in (29), one gets:

$$\begin{aligned} \ln \psi_{(a,b)}(u_0, s) &= \left[\frac{(n+1)^2}{N} \frac{1}{x^N} + \sum_{i=1}^{N-1} \frac{P_i(a,b)}{N-i} \frac{1}{x^{N-i}} \right. \\ &\quad \left. - ((n-2m-1) + P_N(a,b)) \ln |x| \right]_s^{F_+(s)} + O(1). \end{aligned}$$

Writing $F_+(s) = s\bar{F}_+(s)$, $\bar{F}_+(0) \neq 0$, this leads to:

$$\frac{(n+1)^2}{N} \left(\frac{1}{\bar{F}_+(s)^N} - 1 \right) + \sum_{i=1}^{N-1} \frac{P_i(a,b)}{N-i} \left(\frac{1}{\bar{F}_+(s)^{N-i}} - 1 \right) s^i = O(s^N)$$

as $s \rightarrow 0$, or

$$\sum_{i=0}^{N-1} d_i(a,b) (\bar{F}_+(s)^{i-N} - 1) s^i = O(s^N), \quad s \rightarrow 0,$$

for some properly chosen functions $d_i(a,b)$ with $d_0(a,b) = \frac{(n+1)^2}{N} \neq 0$. This implies the result. \blacksquare

Combining Lemma's 9 and 10, leads to the following proposition.

Proposition 11 *Consider the regular transition maps $R_+(s, a, b)$ going from Σ_1^0 to σ_+^1 . Then:*

$$R_+(s, a, b) = s + O(s^{2n+2-m}), \quad s \rightarrow 0.$$

8 The reduced difference map

In this section we will introduce a so-called reduced difference map that will be used in finding an upperbound on the number of limit cycles that can perturb from \mathcal{L} .

By Rolle's theorem, the difference map Δ has at most $N + 1$ zeros in a neighbourhood of zero if $\frac{\partial \Delta}{\partial s}$ has at most N zeros for s near zero, multiplicity taken into account. So, by (32), an upperbound on $\text{Cycl}(\bar{X}_{(a,b)}, \mathcal{L}, (\bar{a}, \bar{b}))$ is found by searching the number of solutions of the equation:

$$\frac{\partial H_+}{\partial s}(s, -a^o, a^e, -b^o, b^e) = \frac{\partial H_+}{\partial s}(s, a^o, a^e, b^o, b^e). \quad (35)$$

However from Proposition 8 and the identity in (31), it is clear that $\frac{\partial H_+}{\partial s}(s, a, b)$ is exponentially flat. For removing this exponential flatness, we introduce a smooth map, called a *reduced difference map* $\bar{\Delta}$, in such a way that its zeroes represent the roots of (35) and hence the zeroes of $\frac{\partial \Delta}{\partial s}$.

Theorem 12 *Let $(X_{(a,b)})$ be a family of general Liénard systems like in (1) admitting an unbounded semi-hyperbolic 2-saddle cycle \mathcal{L} for some parameter value (\bar{a}, \bar{b}) .*

Let D_+ be the normalized expression of $\mathcal{D}_+^{(a,b)}$ and R_+, S_+ the expressions of $\mathcal{R}_+, \mathcal{S}_+$ respectively using normalized sections and parametrisations as in Section 7. Let $A(s, a, b)$ be the map defined in (22).

Then the cyclicity $\text{Cycl}(\overline{X}_{(a,b)}, \mathcal{L}, (\overline{a}, \overline{b}))$ of \mathcal{L} inside $(X_{(a,b)})$ is at most one unit higher than the number of positive zeroes of the map:

$$\overline{\Delta}(s, a, b) = \overline{H}_+(s, -a^o, a^e, -b^o, b^e) - \overline{H}_+(s, a^o, a^e, b^o, b^e), \quad (36)$$

with

$$\begin{aligned} \overline{H}_+(s) &= \overline{H}_+(s, a, b) \\ &= s^{N-1} \left(\log S'_+(D_+(R_+(s))) + \log D_+(R_+(s)) + \log A(R_+(s)) \right. \\ &\quad \left. + \log R'_+(s) - (N+1) \log \frac{R_+(s)}{s} \right), \end{aligned} \quad (37)$$

for $s \sim 0$ and $N = 2n + 1 - m$. The map $\overline{\Delta}(s, a, b)$ is smooth and

$$\overline{\Delta}(s, 0, a^e, 0, b^e) \equiv 0.$$

Moreover:

$$\overline{\Delta}(s, a, b) = 2 \sum_{j=0}^{N/2-1} \frac{P_{2j+1}(a, b)}{N-2j-1} s^{2j} + O(s^{N-1}). \quad (38)$$

Proof. Large amplitude limit cycles arising from \mathcal{L} correspond to small positive zeros of the difference map Δ . Moreover an upperbound for the number of such zeros is one unit higher than the number of solutions of equation (35).

It is clear, from (28) and (31), that

$$\begin{aligned} \frac{\partial H_+}{\partial s}(s) &= \frac{\partial H_+}{\partial s}(s, a, b) \\ &= S'_+(D_+(R_+(s))) D_+(R_+(s)) \frac{A(R_+(s))}{s^{N+1}} \left(\frac{s}{R_+(s)} \right)^{N+1} R'_+(s). \end{aligned}$$

The exponential flatness in this formula is caused by $D_+(R_+(s))$, see Proposition 8. To remove it, we introduce:

$$\overline{H}_+(s, a, b) = s^{N-1} \log \left(s^{N+1} \frac{\partial H_+}{\partial s}(s, a, b) \right),$$

leading to expression (37). From Propositions 8 and 11 and the fact that $A(0) > 0$, $S'_+(0) > 0$ and R_+ is smooth, one has:

$$\overline{H}_+(s, a, b) = -\frac{F(s)}{s} - \sum_{i=1}^{N-1} \frac{P_i(a, b)}{N-i} s^{i-1} + P_N(a, b) s^{N-1} \ln s + s^{N-1} \overline{\Phi}(s, a, b), \quad (39)$$

where

$$\begin{aligned} \overline{\Phi}(s, a, b) &= -\tilde{\Phi}(s, a, b) + \log S'_+(D_+(R_+(s))) + \log A(R_+(s)) \\ &\quad + \log R'_+(s) - (N+1) \log \frac{R_+(s)}{s}. \end{aligned}$$

for some $\tilde{\Phi}(s, a, b)$. The further statements now follow from (39) and Theorem 7, inducing:

$$\begin{aligned} & \overline{H}_+(s, -a^o, a^e, -b^o, b^e) - \overline{H}_+(s, a^o, a^e, b^o, b^e) = \\ & 2 \left(\sum_{j=0}^{(N/2)-1} \frac{P_{2j+1}(a, b)}{N-2j-1} s^{2j} \right) + s^{N-1} (\overline{\Phi}(s, -a^o, a^e, -b^o, b^e) - \overline{\Phi}(s, a^o, a^e, b^o, b^e)) \end{aligned}$$

■

9 Cyclicity in full subfamilies of $(X_{(a,b)})$

We are now ready to treat the cyclicity of unbounded semi-hyperbolic 2-saddle cycles inside full subfamilies of (1) as described in the introduction. We will give a proof of Theorems 1, 2, 3, 4 and 5.

Proof of Theorem 1: From Theorem 12, we know

$$\overline{\Delta}(s, a, b) = 2 \sum_{j=0}^{(N/2)-1} \frac{P_{2j+1}(a, b)}{N-2j-1} s^{2j} + \Upsilon(s, a, b),$$

for some smooth function $\Upsilon(s, a, b) = O(s^{N-1})$, $s \rightarrow 0$. By definition of the Lyapunov quantities (7), one now has:

$$\overline{\Delta}(s, a, b) = \frac{2c_1}{N-2i_1-1} s^{2i_1} (1 + O(s^2)) + 2 \sum_{j=i_2}^{(N/2)-1} \frac{P_{2j+1}(a, b)}{N-2j-1} s^{2j} + \Upsilon(s, a, b).$$

By induction on $l \in \{1, \dots, K\}$:

$$\overline{\Delta}(s, a, b) = \sum_{j=1}^K \frac{2c_j}{N-2i_j-1} s^{2i_j} (1 + O(s^2)) + \Upsilon(s, a, b), \quad (40)$$

with $0 \leq i_1 < \dots < i_K \leq (N/2) - 1$.

Now let $(a, b) = (\bar{a}, \bar{b})$ be a value such that $c_1(\bar{a}, \bar{b}) = \dots = c_{k-1}(\bar{a}, \bar{b}) = 0$ and $c_k(\bar{a}, \bar{b}) \neq 0$ for some $1 \leq k \leq K$. Applying a standard division-derivation algorithm, based on Rolle's theorem on (40), one obtains (i).

We are left with proving (ii). By assumption $(a, b) \mapsto c(a, b)$, with $c(a, b) = (c_1(a, b), \dots, c_K(a, b))$, is a submersion. Therefore one can take, if necessary, a $\lambda(a, b)$ such that $(a, b) \mapsto (c(a, b), \lambda(a, b))$ is a diffeomorphism. Moreover we define $\tilde{\Delta}(s, c, \lambda)$ such that $\overline{\Delta}(s, a, b) = \tilde{\Delta}(s, c(a, b), \lambda(a, b))$ and $\tilde{\Upsilon}(s, c, \lambda)$ such that $\Upsilon(s, a, b) = \tilde{\Upsilon}(s, c(a, b), \lambda(a, b))$. By assumption $\overline{\Delta}(s, \bar{a}, \bar{b}) = \tilde{\Delta}(s, 0, \bar{\lambda}) = 0$, for $\bar{\lambda} = \lambda(\bar{a}, \bar{b})$, such that, using (40), also $\tilde{\Upsilon}(s, 0, \bar{\lambda}) = 0$.

We now prove that $\tilde{\Delta}(s, c, \lambda)$ has at most $K - 1$ zeros near $s = 0$ for $(c, \lambda) \sim (0, \bar{\lambda})$ leading to (ii).

Applying Taylor's theorem on \tilde{Y} with respect to $c_j = 0$ one easily gets, by induction on $j \in \{1, \dots, K\}$, the existence of C^∞ functions $\bar{\Phi}_j$ such that

$$\tilde{Y}(s, c, \lambda) = \sum_{j=1}^K c_j \bar{\Phi}_j(s, c_j, \dots, c_K, \lambda),$$

for $s \sim 0$, $c \sim 0$ and $\forall j : \bar{\Phi}_j(s, c_j, \dots, c_K, \lambda) = O(s^{N-1})$. Substitution in (40) leads to

$$\tilde{\Delta}(s, c, \lambda) = \sum_{j=1}^K c_j \Phi_j(s, c_j, \dots, c_K, \lambda), \quad (41)$$

with $\Phi_j(s, c_j, \dots, c_K, \lambda) = \frac{2}{N-2i_j-1} s^{2i_j} + O(s^{2i_j+2})$, $s \rightarrow 0$.

For values $c = 0$, one has by assumption a center near infinity and therefore no large amplitude limit cycles. So to prove that $Y_{(a,b)}$ has at most K large amplitude limit cycles for $(a, b) \sim (\bar{a}, \bar{b})$, one can restrict to values $(c, \lambda) \sim (0, \bar{\lambda})$ for which $c \neq 0$. Therefore one can assume that $\forall (c, \lambda) \sim (0, \bar{\lambda}) : \exists ! \gamma = \gamma(c)$ with $\max\{|\gamma_1|, \dots, |\gamma_K|\} = 1$ such that $c_i = \rho \gamma_i$, $1 \leq \forall i \leq K$ for some $\rho = \rho(c) \in [0, \rho^*]$, $\rho^* > 0$. From (41) it follows:

$$\tilde{\Delta}(s, c, \lambda) = \rho \sum_{j=1}^K \gamma_j \Phi_j(s, c_j, \dots, c_K, \lambda). \quad (42)$$

Now the zeros of $\tilde{\Delta}(s, c, \lambda)$ near $s = 0$ and for $(c, \lambda) \sim (0, \bar{\lambda})$ correspond to the zeros of the following map near $s = 0$:

$$\Lambda(s, \gamma, \rho, \lambda) = \sum_{j=1}^K \gamma_j \Phi_j(s, \rho \gamma_j, \dots, \rho \gamma_K, \lambda),$$

for $\lambda \sim \bar{\lambda}$ and $\max\{|\gamma_1|, \dots, |\gamma_K|\} = 1$. Now let $(\gamma^0, \rho^0) \in S_m \times [0, \rho^*]$, where S_m denotes the unit sphere for the maximum norm on \mathbb{R}^K . There certainly exists $1 \leq k \leq K$ such that $\gamma_1^0 = \dots = \gamma_{k-1}^0 = 0$ and $\gamma_k^0 \neq 0$. Applying a division-derivation algorithm on $\Lambda(s, \gamma, \rho, \lambda)$ one finds a neighbourhood $V(\gamma^0, \rho^0)$ of $s = 0$ and a neighbourhood $W(\gamma^0, \rho^0)$ of $(\gamma^0, \rho^0, \bar{\lambda})$ such that $\Lambda(s, \gamma, \rho, \lambda)$ has at most $k - 1 \leq K - 1$ zeros for $(s, \gamma, \rho, \lambda) \in V(\gamma^0, \rho^0) \times W(\gamma^0, \rho^0)$. By compactness of $S_m \times [0, \rho^*]$ one can suppose that the neighbourhood $V(\gamma^0, \rho^0)$ is independent of (γ^0, ρ^0) . ■

Theorem 2 is a direct corollary of Theorem 1. Conditions (i) and (ii) imply rather easy expressions of the leading large amplitude Lyapunov quantities enabling us to obtain the requested cyclicity. Let us first, for simplicity, specify the families considered in Theorem 2. In case $m \leq n$ the full sub-

families are given by:

$$\begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^n a_i x^{n+1-i} \right), \\ \dot{y} &= - \left(x^m + \sum_{i=1}^{m-1} b_i x^{m-i} \right), \end{cases} \quad (43)$$

where for each $2i + 1$ either a_{2i+1} or b_{2i+1} is considered to be zero.

In case $m > n$, one considers families of the form

$$\begin{cases} \dot{x} &= y - \left(x^{n+1} + \sum_{i=1}^{2n-m} a_i x^{n+1-i} + \sum_{i=k_1}^{k_2} a_{2i} x^{n+1-2i} \right), \\ \dot{y} &= - \left(x^m + \sum_{i=1}^{2n-m} b_i x^{m-i} + \sum_{i=k_1}^{k_3} b_{2i} x^{m-2i} \right), \end{cases} \quad (44)$$

with the indices (k_1, k_2, k_3) defined such that $2n - m = 2k_1 - 1, n = 2k_2 + 1, m = 2k_3 + 1$ and where for each $2i + 1$ either a_{2i+1} or b_{2i+1} is considered to be zero.

It is easily seen now that Theorems 3 and 4 are special cases of Theorem 2. We can only treat the full subfamilies with odd friction term or odd forcing term in full generality if $m \leq n$. If $m > n$ one has to impose a_{2i+1} and b_{2i+1} to be zero for $2i + 1 > 2n - m$ as in (44).

Proof of Theorem 2: Consider first the case where $(Y_{(a,b)})$ is a full subfamily such that $a^o = b^o = 0$. Then, by Theorem 12, $\bar{\Delta}(s, a, b) \equiv 0$ such that there are clearly no large amplitude limit cycles $\forall (a, b)$.

Suppose now $(a^o, b^o) \neq (0, 0)$. Denote by $i_j, j \in \{1, \dots, K\}$ the indices for which $(a_{2i_j+1}, b_{2i_j+1}) \neq (0, 0)$. Then from Theorem 12, one has $\forall k \leq K$:

$$\bar{c}_k(\bar{a}, \bar{b}) := c_k(\bar{a}, \bar{b}) \Big|_{c_1(\bar{a}, \bar{b}) = \dots = c_{k-1}(\bar{a}, \bar{b}) = 0} = \begin{cases} (n+1)(m-2i_k)\bar{a}_{2i_k+1}, \text{ or} \\ -(n+1)^2\bar{b}_{2i_k+1}. \end{cases}$$

Therefore from Theorem 1 (i) one has $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq K$ or $c_1(\bar{a}, \bar{b}) = \dots = c_K(\bar{a}, \bar{b}) = 0$. One now easily verifies that $\forall 1 \leq k \leq K, (a, b) \mapsto (c_1(a, b), \dots, c_{k-1}(a, b))$ and $(a, b) \mapsto \bar{c}_k(a, b)$ are submersions at (\bar{a}, \bar{b}) implying that $(a, b) \mapsto (c_1(a, b), \dots, c_K(a, b))$ is a submersion too. Furthermore $c_1(\bar{a}, \bar{b}) = \dots = c_K(\bar{a}, \bar{b}) = 0$ implies $\bar{a}^o = \bar{b}^o = 0$ such that $\bar{\Delta}(s, \bar{a}, \bar{b}) = 0$ (Theorem 12). The result now follows from Theorem 1 (ii). ■

Finally we treat an example where the coefficients among a^o and b^o occur in non-trivial pairs inside the full subfamily.

Proof of Theorem 5: We are dealing with a full subfamily of Liénard

systems of type (3, 5). We will show that one can take $K = 2$ in Theorem 1 and use the exact expressions of P_1 and P_3 obtained in Section 5, (27). Indeed for the first two large amplitude Lyapunov quantities, one finds:

$$c_1(a, b) = 18a_1 - 36b_1, c_2(a, b) = a_1(10 + \frac{7}{9}a_1^2).$$

So $c_2(a, b) = 0$ if and only if $a_1 = 0$. Therefore $c_1(\bar{a}, \bar{b}) = c_2(\bar{a}, \bar{b}) = 0$ if and only if $(Y_{\bar{a}, \bar{b}})$ has a center near infinity. So Theorem 1 (i) implies that if $\bar{a}_1 \neq 0$ one has $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq 2$; and if $\bar{a}_1 = 0, \bar{b}_1 \neq 0$ one has $Cycl((Y_{(a,b)}), \mathcal{L}_{(\bar{a}, \bar{b})}, (\bar{a}, \bar{b})) \leq 1$. If $\bar{a}_1 = \bar{b}_1 = 0$, clearly $c_1(\bar{a}, \bar{b}) = c_2(\bar{a}, \bar{b}) = 0$ and $(a_1, b_1) \mapsto (c_1(a_1, b_1), c_2(a_1, b_1))$ is a submersion at $(a_1, b_1) = (0, 0)$. The conditions in Theorem 1 (ii) are then clearly satisfied implying the result. ■

References

- [1] M. Caubergh, F. Dumortier, *Hilbert's 16th problem for classical Liénard equations of even degree*, Journal of Differential Equations, Vol. 244, no.6, 1359-1394, 2008.
- [2] F. Dumortier, *Compactification and desingularisation of spaces of polynomial Liénard equations*, Journal of Differential Equations, Vol. 224, no.2, 296-313, 2006.
- [3] F. Dumortier, R. Roussarie, *Smooth normal linearization of vector fields near lines of singularities*, preprint.
- [4] F. Dumortier, C.Herssens, *Polynomial Liénard equations near infinity*, Journal of Differential Equations, Vol. 153, no. 1, 1-29, 1999.