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A web-based tool to design and analyze single- and double-stage acceptance sampling plans

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ABSTRACT

Acceptance sampling plans are used to determine whether production lots can be accepted or rejected. Existing tools only provide a limited functionality for the two-point design and the risk analysis of such plans. In this article, a web-based tool is presented to study single- and double-stage sampling plans. In contrast to existing solutions, the tool is an interactive applet that is freely available. Analytic properties are derived to support the development of search strategies for the design of double-stage sampling plans that are more efficient and accurate in comparison with existing routines. Several case studies are presented.

Introduction

Acceptance sampling is concerned with the design and implementation of sampling plans to inspect incoming or outgoing production lots. A typical application of acceptance sampling is the inspection by a company of a shipment of items from a supplier. These items are often components or raw material used in the company's manufacturing process. A random sample is taken from the lot, and based on the inspection of some quality characteristic a decision is made to accept or reject the complete lot. The process of making this decision is also termed lot sentencing. Acceptance sampling plans can be classified according to the type of variables that are measured. Quality features that are measured on a numerical scale are used in variables sampling plans, while features that classify items as defective or non-defective lead to attributes sampling plans.

The use of acceptance sampling plans for inspection by attributes dates back to the seminal work of Dodge and Romig (1941). The basic concepts and models of variables sampling plans were introduced by Jennett and Welch (1939). During the past two decades of the 20th century, the research interest in acceptance sampling decreased. The use of acceptance sampling plans

has been criticized as they are described as one-shot deals to test whether a production lot is conform to specifications without giving any feedback into either the production process or engineering design that would be necessarily for quality improvement (Montgomery 2013). However, acceptance sampling is still playing an important role in modern industrial environments and there has been a resurgence of interest in this field in the 21st century (Collani and Göb 2008). First, in many cases the producer has become increasingly removed from the consumer, not only by distance but also by language, culture and governmental differences (Schmueli 2016). There is a need for methods to keep generating pressure on suppliers in order that they maintain and improve the quality in their goods. Second, acceptance sampling plans are able to limit the risk for accepting lots of poor quality. In many applications the quality of incoming goods affect the efficiency of the production and the quality of the end product. For instance, for an egg processing company the freshness of the incoming eggs is essential to minimize drop-out during peeling of boiled eggs as will be discussed in more detail in case study II, presented later in this article. Other examples include the microbiological inspection of incoming goods in food industry for food safety and quality (Santos Fernãndez 2016). Finally,

acceptance sampling can also be adapted to other verification problems as, for example, the verification of probabilistic design requirements using Monte Carlo simulation (White et al. 2009).

Single sampling plan (SSP) and double sampling plan (DSP) are among the most widely used acceptance sampling plans. In the procedure of an SSP plan a decision to accept or reject the lot is based on the result of one sample. The procedure of a DSP plan allows to take a second random sample from the lot when the information from the first sample raises too much doubt (Montgomery 2013). Sampling, however, involves the risk that the sample will not adequately represent an entire lot. There are two types of risk associated to each sampling plan: the producer's (or supplier's) risk and the consumer's (or customer's) risk. The producer's risk reflects the probability to reject an acceptable lot that is defined as a lot with a proportion nonconforming of at most an acceptable quality level p_{AQL}. The customer's risk reflects the probability to accept an unacceptable lot that is defined as a lot with a proportion nonconforming of at least a rejectable quality level p_{RQL}. These risks are analytically studied by the operating characteristic (OC) curve which shows the probability of accepting a lot given various proportions nonconforming. A problem frequently encountered in quality control is the two-point design of a sampling plan where one is interested in the determination of sampling plans that reduce the producer's and consumer's risk below some predefined levels α and β, respectively (Taylor 1997). For this reason, several tables have been presented in the literature to select sampling plans (Duarte and Saraiva 2013; Sommers 1981). Commonly used tables are included in international standards as the ISO standards (and their ANSI/ASQC/BS or other counterparts (Neubauer and Luko 2012, 2013)). Such tables, however, are restricted to a limited number of values for p_{AQL}, p_{RQL}, α and β and are not accompanied with user-friendly tools to evaluate the individual sampling plans regarding the risks associated with sampling error.

Furthermore, a limited number of computer programs are available to study sampling plans and only few offer the ability to design DSP as well as SSP plans. Commercial programs, for example, Minitab, Inc. (2018), only allow the two-point design of an SSP plan. Also, easy-to-use Excel sheets are available to study SSP plans (Bertoni 2016). Kiermeier (2008) developed an R-package 'Acceptance Sampling' that allows to calculate OC-curves of SSP and DSP plans, but the functionality of a two-point design is limited to that of an SSP plan. Furthermore, Cheng and Chen (2007) developed a computer program that is based

on an evolutionary algorithm (Zelinka 2015) to design attributes DSP plans. Its interface, however, requires the interpretation of several parameters and its implementation is outdated.

The design of two-point sampling plans with an OC-curve that passes approximately through two designated points $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β) can be described by a system of algebraic equations. Since such system does not have a closed from solution for SSP and DSP plans, algorithms have been developed to search for feasible combinations of the parameters of the plan that ensure a producer's and consumer's risk below the predefined levels α and β, respectively. Existing routines to design two-point sampling plans can be classified into two main approaches: (i) search routines that perform an exhaustive search to find appropriate combinations of the parameters of a two-point sampling plan and (ii) optimization procedures where some cost function (e.g. the sample size or average sample number (ASN)) is minimized subject to the constraints induced by the two-point method. Optimization procedures have the advantage to be applicable to other types of sampling plans, for example, multiple dependent sampling plans (Balamurali and Jun 2007). However, convergence may not be guaranteed with such approaches and finding appropriate starting values may not be evident for the user (Balamurali and Usha 2013; Duarte and Saraiva 2013). The focus in this article is on search routines that do not depend on starting values and whose implementation will lead to a user-friendly tool to design and analyze DSP plans as well as SSP plans.

Early development of search routines for the design of sampling plans were performed with the programing language FORTRAN (Chow et al. 1972; Hailey 1980). A general algorithm to develop two-point SSP plans for inspection by attributes was introduced by Hailey (1980). For variables SSP plans computational formulas are available (Schilling and Neubauer 2009). The two-point design of DSP plans, however, is more complex as more parameters are involved (two sample sizes n_1 and n_2 and two acceptance numbers c_1 and c_2 to test sample results). To reduce the number of possible combinations of parameters, a fixed relationship between the sample sizes n_1 and n_2 is often assumed. The most common constraint that has been used is to require that n_2 is a multiple of n_1, that is, $n_2 = r \cdot n_1 (r \in \mathbb{N}_0)$ (Montgomery 2013). For attributes inspection, previous developed search routines have assumed a Poisson distribution (Chow et al. 1972). In addition, additional constraints on the acceptance numbers (c_1, c_2) have been studied, for example, $c_2 \geq (r + 1)c_1$ or $(c_1, c_2) =$

(0, 1) (Newman and Yu 2018; Olorunniwo and Salas 1982; Vijayaraghavan 2007). For variables inspection, Sommers (1981) has developed a table of DSP plans with a minimum ASN at p_{AQL}. Unique DSP plans have been obtained by minimizing other cost functions as well, for example, the maximum ASN (Krumbholz and Rohr 2009; Vangjeli 2012).

In this article, we develop search routines for the design of attributes and variables DSP plans that minimize the ASN at p_{AQL}. For this purpose, it is assumed that n_2 is a multiple of n_1. Analytic properties of DSP plans are derived that are given a mathematical proof and that will support the development of the search routines. For attributes inspection, the number of nonconforming items in a lot is modeled by a binomial distribution. It is shown that several constraints hold on the parameters of an attributes DSP plan reducing the number of possible combinations of parameters that have to be considered during an exhaustive search routine (Olorunniwo and Salas 1982). For variables inspection, the measurement data are modeled using a normal distribution with known standard deviation σ. The search routine to design variables DSP plans is shown to result in more accurate minimum values of the ASN when compared to the routines introduced by Sommers (1981). Furthermore, search routines are implemented in a user-friendly and interactive web tool to design and analyze SSP and DSP plans. Unlike previous developed solutions, the tool is an interactive applet that is easy and freely accessible and that supports the two-point design of DSP plans for inspection by attributes as well as variables. Several case studies from food industry are discussed to illustrate the use of the tool.

Note that the study of variables inspection plans is restricted to the case where one specification limit is defined and where measurements are drawn from a normal distribution with known variance. Extensions of the properties and design to the case of double specification limits or to the case where quality features are studied that follow other probability distributions (e.g., an exponential distribution) can be a subject of further research. The remainder of the article is structured as follows. Firstly, the reader is introduced to the necessary terminology and notations of acceptance sampling. Subsequently, several analytic properties of the OC-curves of DSP plans are derived. Moreover, search routines to develop two-point DSP plans are presented. Next, these routines, together with procedures to design SSP plans, are implemented in a web-based tool that is illustrated by several case studies. Finally, a conclusion is made.

Background on acceptance sampling

Firstly, an introduction is given to the risk analysis of acceptance sampling plans using OC-curves. Next, a general background on SSP and DSP plans is given.

Risk analysis and OC-curves

Sampling plans are subject to sampling error and therefore induce risks. A fundamental tool to describe the risks associated to a sampling plan $\mathcal{P}$ is the OC-curve $p \mapsto \phi(p, \mathcal{P})$ which relates the proportion nonconforming present in the lot to the probability of acceptance of a lot using the plan $\mathcal{P}$. The statistical design of a sampling plan is based on two specific points on the OC-curve:

(1) The producer's risk point denoted as $(p_{AQL}, 1 - \alpha)$. The acceptable quality level p_{AQL} represents the maximal proportion nonconforming that is allowed to accept a lot. Therefore, from the producer's side, a lot with $p < p_{AQL}$ should be accepted with a high probability $1 - \alpha$. The risk α of rejecting an acceptable lot with $p < p_{AQL}$ is termed the producer's risk.

(2) The consumer's risk point denoted as (p_{RQL}, β). The rejectable quality level p_{RQL} represents the minimal proportion nonconforming that is required to reject the lot. From the consumer's side a lot with $p > p_{RQL}$ is not acceptable and should be rejected with a high probability $1 - \beta$. The risk β to accept a rejectable lot with $p > p_{RQL}$ is termed the consumer's risk.

Ideally the quality engineer would like to set $p_{AQL} = p_{RQL}$ to design a sampling plan that would accept all lots with $p < p_{AQL}$ and reject all lots with $p > p_{RQL}$. However, such ideal plan can only be realized by 100% inspection, if the inspection would be error-free. Therefore, one sets $p_{RQL} > p_{AQL}$ and a sampling plan is designed such that the OC-curve passes approximately through the points $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β):

$$\begin{cases} \phi(p_{AQL}, \mathcal{P}) & \geq & 1 - \alpha, \\ \phi(p_{RQL}, \mathcal{P}) & \leq & \beta. \end{cases} \quad [1]$$

The ideal OC-curve of a 100% inspection plan can be approached by increasing the sample size(s) of the sampling plan. The slope of the OC-curve is a measure for the discriminating power of the plan. In the next section, we will illustrate these principles on SSP and DSP plans.

Single sampling plans

An SSP plan for inspection by attributes is a procedure in which a decision is made to accept or reject a lot of size N based on the number of nonconforming items of a random sample of size $n < N$ taken from it. Such a sampling plan is defined by two integers (n, c), with $c < n$, and where n denotes the sample size and c denotes the maximal number of nonconforming items c that are allowed in order to accept the lot under inspection (the so-called acceptance number).

When the lot size is large ($\frac{n}{N} < 0.1$), the number of nonconforming items D that is found in samples of size n drawn from the lot follows approximately a binomial distribution, that is, $D \sim B(n, p)$ where p denotes an (unknown) lot fraction nonconforming p. The OC-curve $p \mapsto \phi_a^{ssp}(p, n, c)$ of an SSP-(n, c) plan is given by:

$$\phi_a^{ssp}(p, n, c) = P(D \leq c)$$
$$= \sum_{j=0}^{c} \frac{n!}{j!(n-j)!} \; p^j (1-p)^{n-j} \qquad [2]$$

and a statistical design is a solution of the system of Eq. [1] that is given by:

$$\begin{cases} \sum_{j=0}^{c} \frac{n!}{j!(n-j)!} \; p_{AQL}^j (1-p_{AQL})^{n-j} \geq 1-\alpha, \\ \sum_{j=0}^{c} \frac{n!}{j!(n-j)!} \; p_{RQL}^j (1-p_{RQL})^{n-j} \leq \beta. \end{cases} \qquad [3]$$

Figure 1(a) shows how the OC-curves of attributes SSP plans change when the sample size n and the acceptance number c change. The OC-curve becomes more like the idealized OC-curve when the sample size n increases. Plans with a lower acceptance number c provide more discriminating power at lower proportions nonconforming. Minimal sample sizes

can be achieved by considering zero acceptance number sampling plans $(n, 0)$ (Hahn 1974).

When a continuous variable X in a lot of size N is inspected, one can use an SSP plan for inspection by variables. Such plan assumes that the variable X follows a normal distribution $N(\mu, \sigma^2)$ with mean μ and a known standard deviation σ. When an upper-specification limit U on the variable X is defined, a single lot is accepted under the condition $\frac{U - \bar{X}_n}{\sigma} \geq k$, where k is a continuous acceptance constant indicating the minimal standardized distance between the sample mean $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ and the upper-specification limit U. Similarly, when a lower-specification limit L is used, the condition is given by $\frac{\bar{X}_n - L}{\sigma} \geq k$. In both cases the OC-curve for a variables SSP-(n, k) plan is defined by:

$$\phi_v^{ssp}(p, n, k) = P\left(Z \leq \sqrt{n}(Z_p - k)\right), \qquad [4]$$

where $Z_p = \Phi^{-1}(1-p)$ and Φ denotes the cumulative distribution function of a variable Z following a standard normal distribution $N(0, 1)$.

Figure 1(b) shows how the OC-curves of variables SSP plans change as the sample size n and the acceptance constant k change. A higher acceptance constant implies a higher discriminative power at lower proportions nonconforming. Larger sample sizes result in a better protection of producers and consumers and lead to OC-curves that become more like the idealized OC-curve.

In case the measurements show a strong deviation from normality, cautionary is required in applying formula [4]. Alternative variables SSP $-(n, k)$ plans exist for exponential, gamma and Weibull distributions. An overview is given by White and Johnson (2013).
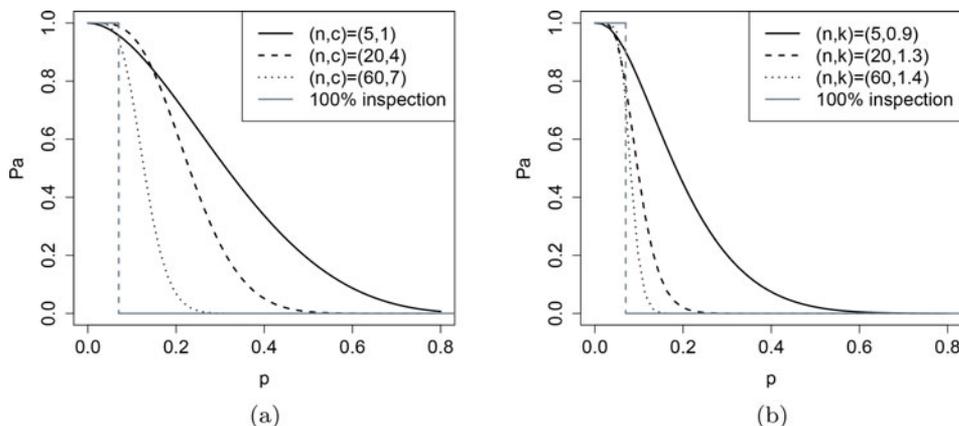

Figure 1. OC-curves of several sampling plans where P_a denotes the acceptance probability: (a) attributes sampling plans and (b) variables sampling plans. The idealized OC-curves corresponding with complete inspection are shown in gray.

Double sampling plans

A DSP plan allows to take a second random sample when the information from the first sample raises to much doubt about the decision whether to accept or reject the lot. When a second sample is taken, the information from the first and second sample is combined in order to decide whether to accept or reject the lot.

Following Montgomery (2013), a DSP plan for attributes inspection is defined by four parameters (n_1, n_2, c_1, c_2) with $c_1 < c_2$ and operates as follows:

(1) **Stage I.** A random sample of size n_1 is taken from the lot. If the number of nonconforming items D_1 that is found in the first sample does not exceed the first sample acceptance number c_1, the lot is accepted; If D_1 exceeds the second sample acceptance number c_2, the lot is rejected.

(2) **Stage II.** If $c_1 < D_1 \leq c_2$, a second sample of size n_2 is taken from the lot. In this case the total number of nonconforming items D_1 and D_2 that is found in stage 1 and stage 2, respectively, is considered. When this total amount $D = D_1 + D_2$ does not exceed the acceptance number c_2, the lot is accepted; Otherwise, it is rejected.

The OC-curve of a DSP-(n_1, n_2, c_1, c_2) plan is given by:

$$\phi_a^{dsp}(p, n_1, n_2, c_1, c_2) = P(D_1 \leq c_1)$$
$$+ \sum_{j=c_1+1}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j). \quad [5]$$

In contrast to an SSP plan, the total number of inspected items is not constant, but depends on the number of nonconforming items found in the first sample. The probability of drawing a second sample varies with the proportion nonconforming that determines the ASN. The ASN of a DSP plan is determined by:

$$\text{ASN} = n_1 P_I + (n_1 + n_2)(1 - P_I), \quad [6]$$

where P_I is the probability that the lot is accepted or rejected on the first sample, that is, $P_I = P(D_1 \leq c_1) + P(D_1 > c_2)$. The ASN curve shows the ASN as a function of the lot fraction nonconforming p.

For variables inspection, we follow Sommers (1981) and define a DSP plan by four parameters (n_1, n_2, k_1, k_2) with $k_1 < k_2$ and with the following operating procedure:

(1) **Stage I.** A random sample of size n_1 is taken from the lot. When the standardized difference

between the sample mean and the lower (or upper) specification limit, that is,

$$V = \frac{\bar{X}_n - L}{\sigma} \quad \left(\text{or } V = \frac{U - \bar{X}_n}{\sigma}\right)$$

exceeds the second sample acceptance constant k_2, the lot is accepted; When it does not exceed the first sample acceptance constant k_1, the lot is rejected;

(2) **Stage II.** When $k_1 < V \leq k_2$, a second sample of size n_2 is taken. In this case the mean of all measurements $\bar{X}_T$ from the samples taken in stages 1 and 2 is considered. When the standardized difference between this overall mean and the lower (or upper) specification limit is lower than the constant k_1, the lot is rejected. Otherwise, it is accepted.

The OC-curve of a DSP-(n_1, n_2, k_1, k_2) plan is given by:

$$\phi_v^{dsp}(p, n_1, n_2, k_1, k_2) = P\left(Z \leq \sqrt{n_1}(Z_p - k_2)\right) + P_2, \quad [7]$$

where $Z \sim N(0,1)$ and P_2 is a cumulative probability associated to a bivariate normal distribution $(W_1, W_2) \sim N(0, \Sigma)$ with:

$$\Sigma = \begin{pmatrix} 1 & \sqrt{\dfrac{n_1}{n_1 + n_2}} \\ \sqrt{\dfrac{n_1}{n_1 + n_2}} & 1 \end{pmatrix}, \quad [8]$$

and

$$P_2 = P\left(W_1 \leq \sqrt{n_1 + n_2}(Z_p - k_1), \right.$$
$$\left. \sqrt{n_1}(Z_p - k_2) \leq W_2 \leq \sqrt{n_1}(Z_p - k_1)\right). \quad [9]$$

The dependency between the variables W_1 and W_2 in the expression of ϕ_v^{dsp} is due to the dependency of the result in the second stage and the result in the first stage. Indeed, the decision in the second stage is based on $\bar{X}_T = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2}{n_1 + n_2}$ and therefore depends on the result of the sample taken in the first stage (Sommers 1981).

The expression for the ASN curve is based on [6]. The probability P_I is now given by:

$$P_I = P\left(Z \leq \sqrt{n_1}(Z_p - k_2)\right) - P\left(Z \leq \sqrt{n_1}(Z_p - k_1)\right) \quad [10]$$

Search procedures for double sampling plans

In this section, we provide the reader with algorithms to design DSP plans. A statistical design relies on a solution of the system of Eq. [1] that ensures that the OC-curve passes approximately through the two

points $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β). As the parameters of DSP plans are non-negative integers, one can rely on mixed integer non-linear programing techniques. However, convergence may not be guaranteed by such approaches and finding appropriate starting values may not be evident for the user (Duarte and Saraiva 2013). Alternatively, routines can be developed that list all possible parameter combinations of DSP plans that solve [1]. To keep the computation time limited, an additional constraint is chosen on the sample sizes of the plans requiring that n_2 is a multiple of n_1. A unique sampling plan is returned by minimizing the ASN at p_{AQL}.

Design of two-point attributes double sampling plans

Let us first recall some properties of the design of an SSP-(n, c) plan. Geometrically, a two-point SSP-(n, c) plan with an OC-curve that approximately passes through the points $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β) will be situated in a region of the nc-plane consisting of all the points (n, c) that satisfy $n_l(c) \leq n \leq n_u(c)$ with:

$$n_l(c) = \inf\{n | \phi_a^{ssp}(p_{RQL}, n, c) \leq \beta\} \quad \text{and}$$
$$n_u(c) = \sup\{n | \phi_a^{ssp}(p_{AQL}, n, c) \geq 1-\alpha\}, \quad [11]$$

and where n_l and n_u are non-decreasing as a function of c. If $n_u(c) < n_l(c)$, there does not exist a two-point SSP-(n, c) plan for the points $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β). In what follows, c^* denotes the minimal acceptance number such that $n_l(c^*) \leq n_u(c^*)$. Setting $n^* = n_l(c^*)$ leads to the two-point SSP-(n^*, c^*) plan with minimum sample size n^* (Luca 2018).

In the following property, the effect is studied of changes in sample sizes or acceptance numbers on the OC-curve of a two-point DSP-(n_1, n_2, c_1, c_2) plan. A proof is given in Appendix A. The property will support the development of an efficient search procedure for the design of a two-point DSP plan.

Property 1. *Consider a DSP-(n_1, n_2, c_1, c_2) plan and it's corresponding OC-curve:*

$$p \mapsto \phi_a^{dsp}(p, n_1, n_2, c_1, c_2).$$

(i) $\phi_a^{dsp}(p, n_1, n_2, c_1, c_2)$ *is strictly increasing as a function of c_1 and c_2.*

(ii) $\phi_a^{dsp}(p, n_1, n_2, c_1, c_2)$ *is strictly decreasing as a function of n_1 and n_2.*

(iii) *The OC-curve of a DSP-(n_1, n_2, c_1, c_2) plan is situated between the OC-curves of an*

SSP-(n_1, c_1) *and an SSP-(n_1, c_2) plan:*

$$\forall p : \phi_a^{ssp}(p, n_1, c_1) \leq \phi_a^{dsp}(p, n_1, n_2, c_1, c_2)$$
$$\leq \phi_a^{ssp}(p, n_1, c_2).$$

In particular, if $c = c_1 = c_2$, the DSP-(n_1, n_2, c_1, c_2) plan is equivalent with an SSP-(n_1, c) plan and:

$$\forall p : \phi_a^{dsp}(p, n_1, n_2, c_1, c_2) = \phi_a^{ssp}(p, n_1, c_1).$$

(iv) *Consider the SSP-(n^*, c^*) plan with an OC-curve passing through $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β) and with a minimum sample size n^* and an acceptance number c^*. Several constraints hold on the matching two-point DSP-(n_1, n_2, c_1, c_2) plan with a minimum sample size n_1: $n_1 + n_2 \geq n_l(c_2), c_2 \geq c^*$ and $n_1 \leq n^*$ for $c_1 \leq c^*$. Furthermore, as $c_1 \uparrow c^*$ (i.e. c_1 approaches c^* from the left-hand side) the matching two-point DSP-(n_1, n_2, c_1, c_2) plan with a minimum sample size n_1 will degenerate to the SSP-(n^*, c^*) plan (i.e. $c_1 = c_2 = c^*$ and $n_1 = n^*, n_2 = 0$).*

In Algorithm 1, we present the pseudo-code of a search procedure for a two-point DSP-(n_1, n_2, c_1, c_2) plan with $n_2 = rn_1$ ($r \in \mathbb{N}_0$) and with a minimum ASN such that the OC-curve passes through two specified points $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β). The first step in designing a two-point attributes DSP-(n_1, n_2, c_1, c_2) plan is to search the corresponding SSP-(n^*, c^*) plan with an OC-curve that passes through the two points $(p_{AQL}, 1-\alpha)$ and (p_{RQL}, β). From Property 1 (iv), it follows that an exhaustive search on the parameters (n_1, n_2, c_1, c_2) can be restricted to acceptance numbers (c_1, c_2) satisfying $c_1 < c^* \leq c_2$. The latter restriction leads to a more efficient search when compared to existing procedures that test every combination of the acceptance numbers c_1 and c_2 (Olorunniwo and Salas 1982). Furthermore, because $c_2 \geq c^*$, one finds $n_1 + n_2 \geq n_l(c_2) \geq n^*$ such that $n_1 \geq \lfloor \frac{n^*}{r+1} \rfloor$.

The search strategy of Algorithm 1 (see Figure 2) calculates for each first sample acceptance number $c_1 < c^*$ a corresponding second sample acceptance number $c_2 \geq c^* > c_1$ and sample sizes $n_1 > c_2, n_2 = r \cdot n_1$ that are required to assure that as well the producers as the consumers are protected. For a given c_1, one starts with initialization by considering the sampling plan with n_1 and c_2 as small as possible. When $\hat{\alpha} \leq \alpha$ and $\hat{\beta} \leq \beta$ the while loops between line 10 and line 26 are not entered and the sampling plan $(2, 2, 0, 1)$ is chosen. When the consumer's risk is lower than the requested limit β and the producer's risk is higher than α, an increase of the

Algorithm 1 Design of attributes DSP plans

1: **procedure** DSP4A(p_{AQL}, α, p_{RQL}, β, r)
2:     **find two-point SSP plan** (n^*, c^*)
3:     **initialize vectors** c_1, c_2 and n
4:     $i \leftarrow 1$;
5:     **while** $i \leq c^*$ **do**
6:         $c_{1i} \leftarrow i - 1$; $c_{2i} \leftarrow \max\{c^*, c_{1i} + 1\}$
7:         $n_0 \leftarrow \max\left\{c_{2i} + 1, \left\lfloor \frac{n^*}{r+1} \right\rfloor\right\}$
8:         $\hat{\alpha} \leftarrow 1 - \phi_2^a(p_{AQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
9:         $\hat{\beta} \leftarrow \phi_2^a(p_{RQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
10:         **while** $\hat{\beta} \leq \beta$ and $\hat{\alpha} > \alpha$ **do**
11:             $c_{2i} \leftarrow c_{2i} + 1$;
12:             $\hat{\alpha} \leftarrow 1 - \phi_2^a(p_{AQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
13:             $\hat{\beta} \leftarrow \phi_2^a(p_{RQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
14:         **end while**
15:         **while** $\hat{\beta} > \beta$ **do**
16:             **while** $\hat{\alpha} \leq \alpha$ and $\hat{\beta} > \beta$ **do**
17:                 $\hat{\alpha} \leftarrow 1 - \phi_2^a(p_{AQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
18:                 $\hat{\beta} \leftarrow \phi_2^a(p_{RQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
19:                 $n_0 \leftarrow n_0 + 1$
20:             **end while**
21:             **if** $\hat{\alpha} > \alpha$ **then**
22:                 $c_{2i} \leftarrow c_{2i} + 1$
23:             **end if**
24:             $\hat{\alpha} \leftarrow 1 - \phi_2^a(p_{AQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
25:             $\hat{\beta} \leftarrow \phi_2^a(p_{RQL}, n_0, r \cdot n_0, c_{1i}, c_{2i})$
26:         **end while**
27:         $n_i \leftarrow n_0$; $i \leftarrow i + 1$
28:     **end while**
29:     **return** (n_i, c_{1i}, c_{2i}) with minimum ASN
30: **end procedure**

Algorithm 2 Design of variables DSP plans

1: **procedure** DSP4V(p_{AQL}, α, p_{RQL}, β, r)
2:     **find two-point SSP plan** (n^*, k^*)
3:     **initialize vectors** k_1, k_2 and n
4:     $n_1 \leftarrow n^*$
5:     $k_{11} \leftarrow k^*$; $k_{21} \leftarrow k^*$
6:     $i \leftarrow 2$;
7:     **repeat**
8:         $n_0 \leftarrow n^* - i$; $i \leftarrow i + 1$
9:         $G(k) \leftarrow \phi_1^v(p_{AQL}, n_0, k)$
10:         $k_{1i}^{max} \leftarrow$ **solve** $G(k) = 1 - \alpha$
11:         **repeat**
12:             $k_{1i} \leftarrow k_{1i} - 0.0001$
13:             $H(k) \leftarrow \phi_1^v(p_{RQL}, n_0, r \cdot n_0, k_{1i}, k)$
14:             **if** $H(0)H(5) < 0$ **then**
15:                 $k_{2i} \leftarrow$ **solve** $H(k) = \beta$
16:                 $\hat{\alpha} \leftarrow 1 - \phi_2^v(p_{AQL}, n_0, r \cdot n_0, k_{1i}, k_{2i})$
17:             **end if**
18:         **until** $\hat{\alpha} \leq \alpha$ or $H(0)H(5) > 0$
19:         $n_i \leftarrow n_0$
20:     **until** $i \geq n^*$ or $H(0)H(5) > 0$
21:     **return** (n_i, k_{1i}, k_{2i}) with minimum ASN
22: **end procedure**

Figure 2. Algorithms for the design of two-point DSP plans. The parameters of the procedures determine the producer's and consumer's risk point. A constraint $n_2 = rn_1$ on the sample sizes can be chosen by specifying $r \in \mathbb{N}_0$.

acceptance number c_2 is considered (which increases the consumer's risk but lowers the producer's risk due to Property 1(i)) while keeping the sample size as low as possible (such that the producer's risk is at its minimum) until the producer's risk is lower than α or the customer's risk exceeds β. At the end of the while loop starting at rule 10 of Algorithm 1, either $\hat{\alpha} \leq \alpha$ and $\hat{\beta} \leq \beta$ leading to the desired sampling plan or $\hat{\beta} > \beta$. In the latter, one enters the while loop at rule 15 where the sample size is increased when the producer's risk is lower than α (Property 1(ii)). At rule 20 the desired plan is found or $\hat{\alpha} > \alpha$ such that the acceptance number c_2 has to be increased to reduce the producer's risk. The while loop that started at line 15 continues when $\hat{\beta} > \beta$. Next, c_1 is increased and the while loop over i that started at line 5 restarts a next iteration.

At the end, the procedure has calculated a series of DSP-(n_1, n_2, c_1, c_2) plans with $0 \leq c_1 \leq c^*$ and $n_2 = r \cdot n_1$ and with OC-curves that pass through the two points $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β). Using Eq. [6], the sampling plan with the minimum ASN can be returned.

Remark that Algorithm 1 can be adapted to the hypergeometric case where the lot size N is an additional parameter. Indeed, the dependency on the sample sizes and criteria of the acceptance probability related to a hypergeometric distribution is similar to the dependency on the sample sizes and criteria of the acceptance probability of a binomial distribution such that Property 1 will also hold for the hypergeometric case. Furthermore, other constraints on the sample sizes can be considered by replacing $f(n_0) = r \cdot n_0$ in Algorithm 1 by another function f.

Design of two-point variables double sampling plans

The following property studies the effect of the changes in the parameters of a two-point variables DSP-(n_1, n_2, k_1, k_2) plan on its OC-curve. Due to the
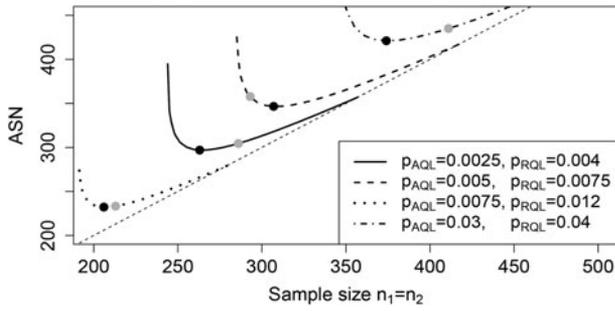
Figure 3. The ASN at p_{AQL} as a function of the sample size n_1 $= n_2$ for several two-point DSP-(n_1, n_2, k_1, k_2) plans that pass through two points $(p_{AQL}, 0.95)$ and $(p_{RQL}, 0.1)$. The dotted identity line correspond to an SSP-(n_1, k_1) where ASN $= n_1$. The gray and black dots correspond to the plans obtained by Sommers (1981) and Algorithm 2, respectively.

scalar nature of the criteria (k_1, k_2), the proof of Property 2 requires a different reasoning than the proof of Property 1 (see Appendix A).

Property 2. *Consider a variables DSP-(n_1, n_2, k_1, k_2) plan and it's corresponding OC-curve:*

$$p \mapsto \phi_v^{dsp}(p, n_1, n_2, k_1, k_2).$$

(i) $\phi_v^{dsp}(p, n_1, n_2, k_1, k_2)$ *is strictly decreasing as a function of k_1 and k_2.*

(ii)   *There exist $p_0, p_1 \in]0, 1[$ such that $\phi_v^{dsp}(p, n_1, n_2, k_1, k_2)$ strictly increases as a function of n_1 and n_2 for $p \in [0, p_0[$ and strictly decreases as a function of n_1 and n_2 for $p \in]p_1, 1]$. In particular, an increase in sample size leads to a decrease in producer's and consumer's risk for $p_{AQL} \in]0, p_0[$ and $p_{RQL} \in]p_1, 1[$.*

(iii)   *The OC-curve of a DSP-(n_1, n_2, k_1, k_2) plan is situated between the OC-curves of an SSP-(n_1, k_1) and an SSP-(n_1, k_2) plan:*

$$\forall p : \phi_a^{ssp}(p, n_1, k_2) \leq \phi_a^{dsp}(p, n_1, n_2, k_1, k_2)$$
$$\leq \phi_a^{ssp}(p, n_1, k_1).$$

If $k_1 = k_2$, the DSP-(n_1, n_2, k_1, k_2) plan is equivalent with an SSP-(n_1, k_1) plan and:

$$\forall p : \phi_v^{dsp}(p, n_1, n_2, k_1, k_2) = \phi_v^{ssp}(p, n_1, k_1).$$

(iv)   *Consider the SSP-(n^*, k^*) plan with an OC-curve passing through $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β) and with a minimum sample size n^* and an acceptance constant k^*. Several constraints hold on the matching two-point DSP-(n_1, n_2, k_1, k_2) plan with a minimum sample size n_1: $k_1 \leq k^* \leq k_2$ and $n_1 \leq n^*$ for $k_2 \geq k^*$. Furthermore, as $k_2 \downarrow k^*$, the matching DSP-(n_1, n_2, k_1, k_2) plan with a minimum sample size n_1 will degenerate*

to the SSP-(n^*, k^*) plan (i.e. $k_1 = k_2 = k^*$ and $n_1 = n^*, n_2 = 0$).

The procedure to calculate a DSP plan for variables inspection is given in Algorithm 2. As with attributes inspection, the procedure starts with the design of an SSP-(n^*, k^*) plan with an OC-curve that passes through two points $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β) such that n^* becomes an upper bound of n_1 (Property 2(iv)). For each $n_1 < n^*$, criteria (k_1, k_2) are calculated that guarantee a producer's risk and a consumer's risk of at most α and β, respectively. For this purpose, an SSP-(n_1, k_1^*) plan is calculated to find an upper bound on k_1 that corresponds to a producer's risk α (Property 2(ii)). Consequently, a grid search is performed to find appropriate values for k_1 and k_2 in the repeat loop starting at rule 7. By lowering k_1, the consumer's risk increases which can be compensated by an increase in k_2 such that the consumer's risk is at most β (Property 2(i)). At the same time, it is checked whether the producer's risk stays below α. When n_1 is chosen too low, the consumer's risk will be too high and no solution won't be found for $k_2 \in [0, 5]$ such that the loop terminates (Property 2(ii)). Choosing a range of $[0, 5]$ for k_2 enables to find solutions for a wide range of values for the parameters of DSP plans (see Table B1 in Appendix B). Finally, the plan is returned with the minimum ASN by use of [6] and [10].

Tabulated single and double sampling plans

To illustrate the use of Algorithms 1 and 2, we present a table of matching SSP and DSP plans for variables and attributes inspection in Appendix B. The SSP plans are calculated using the search routines described by Kiermeier (2008). The DSP plans are calculated by minimizing the ASN at p_{AQL}. The plans are matched such that the OC-curves pass through the two points $(p_{AQL}, 0.95)$ and $(p_{RQL}, 0.1)$.

The DSP plans for inspection by variables are defined following the principles of Sommers (1981). In contrast to the procedure proposed by Sommers (1981), the use of Algorithm 2 resulted in more accurate minima of the ASN at p_{AQL}. To illustrate this, Figure 3 shows the ASN at p_{AQL} as a function of the sample size $n_1 = n_2$ of the DSP-(n_1, n_2, k_1, k_2) plans that pass through the two points $(p_{AQL}, 0.95)$ and $(p_{RQL}, 0.1)$. The curves correspond to several choices of p_{AQL} and p_{RQL}. The gray dots indicate the plans tabulated by Sommers (1981), while the black dots indicate the plans obtained by using Algorithm 2.

Clearly, the use of Algorithm 2 results in sample sizes that correspond more accurately to a minimum of the ASN curve.

A web tool to design and study sampling plans

In this section a web tool is presented to develop and study SSP and DSP plans. The web tool is available at https://www.acceptancesampling.com/ and aims at providing a user-friendly way to develop and analyze two-point sampling plans. The tool is organized in two sheets: one sheet for attributes inspection and one sheet for variables inspection (Figure 4). Routines underlying the statistical computations were implemented in R (R Core Team 2018). The web interface is based on the PHP-language and the HTML-library bootstrap (Duckett 2014; Spurlock 2013). The use of four different types of sampling plans was implemented:

i. SSP plans for inspection by attributes based on the binomial distribution.
ii. SSP plans for inspection by variables based on a normal distribution with a known standard deviation σ.
iii. DSP plans for inspection by attributes based on the binomial distribution.
iv. DSP plans for inspection by variables based on a normal distribution with a known standard deviation σ.

Each sheet consists of a left and a right panel of which the structure and functionality are very similar across the different types of plans to enhance user-friendliness.

Introducing the panels of the web tool

The interface consists of a left panel that shows the parameters of a plan and a right panel that visualizes a plan by the OC-curve or the ASN curve (Figure 4). In particular, the left panel shows the sample sizes and relevant criteria of the sampling plan together with the risks related to sampling error. To increase the ease of interpretability for practitioners the terms supplier's risk and customer's risk are used. A typical application of the tool is the inspection by a company (the customer) of an incoming lot from a supplier. Furthermore, the quality levels can be specified: the acceptable quality level, abbreviated as AQL and the rejectable quality level, abbreviated as RQL. A help page is available at the top right corner to introduce the user to the functionality of the tool and the terminology that is used.

The use of the left panel is implemented in two directions: (i) one can calculate a sampling plan given
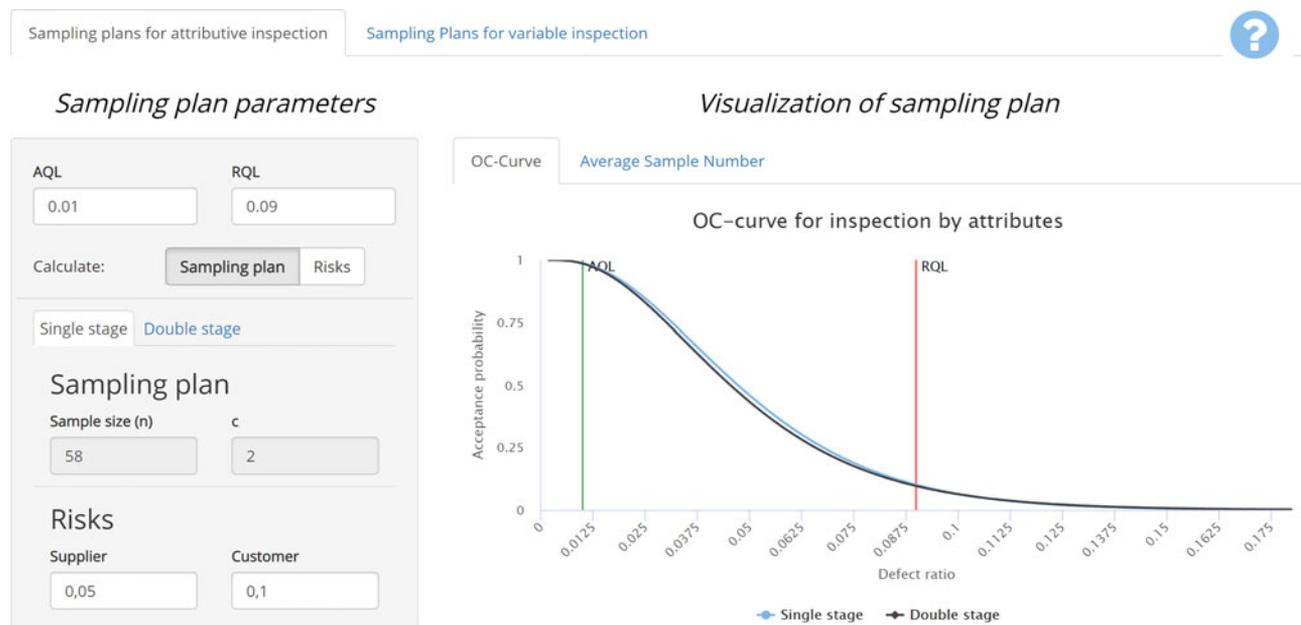
Figure 4. The interface of our web tool that consists of a left panel to set the sampling plan parameters and a right panel to visualize the plan with an OC-curve and an ASN curve. The selection of the main sheet determines the type of inspection: attributes inspection or variables inspection. The left panel allows to implement single-stage and double-stage sampling plans.
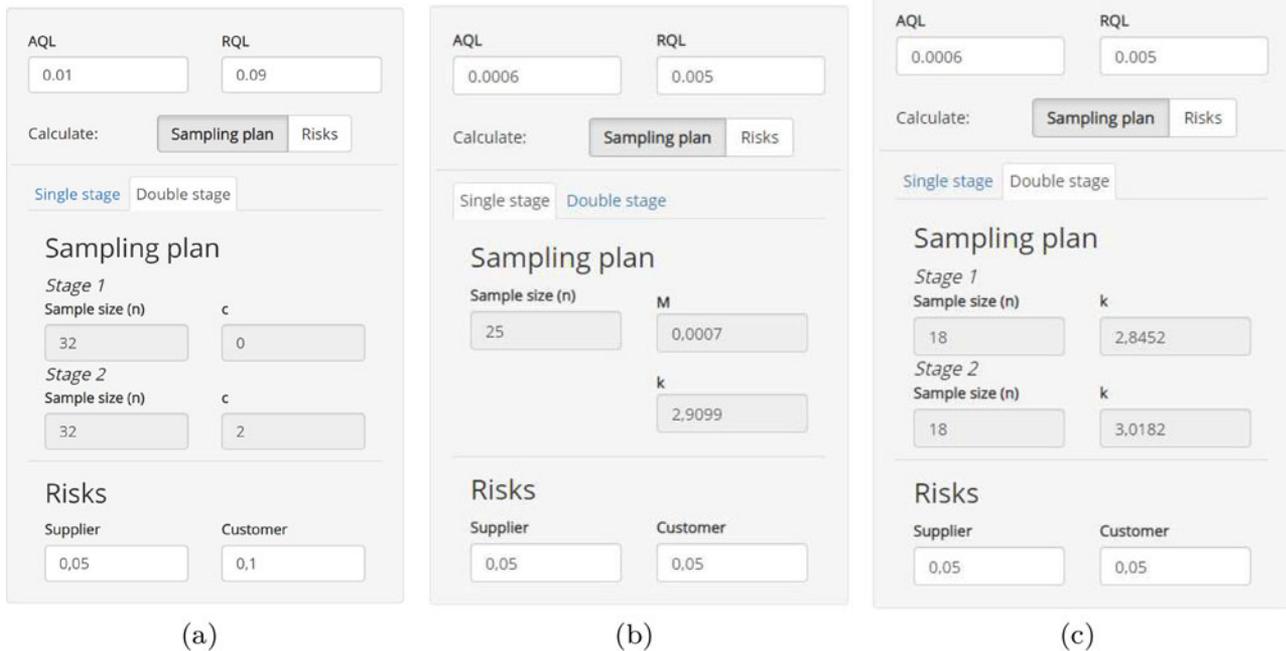
Figure 5. Left panels of the interface: (a) the parameters of a DSP-(n_1, n_2, c_1, c_2) plan for inspection by attributes, (b) the parameters of an SSP-(n, k) plan for inspection by variables, and (c) the parameters of a DSP-(n_1, n_2, k_1, k_2) for inspection by variables. The panel in (a) is part of the main sheet entitled sampling plans for attributes inspection as shown in Figure 4. The panels in (b) and (c) are part of the main sheet entitled sampling plans for variables inspection.

the parameters (p_{AQL}, α) and (p_{RQL}, β) or (ii) the risks α and β are calculated given the sampling plan and the quality levels p_{AQL} and p_{RQL}. The sampling plans are obtained by an implementation of Algorithms 1 and 2. Risks can be calculated using formulas [2], [4], [5], [7], and [9].

The layout of the left panel depends on the type of sampling plan (Figure 5). For a variables SSP-(n, k) plan, lot acceptance can also be described by an upper bound M on an unbiased estimate of the lot fraction nonconforming (the so-called M-method). When the standard deviation is known, this upper bound may be found by $P(Z \geq k\sqrt{\frac{n}{n-1}})$ for $Z \sim N(0, 1)$ (Schilling and Neubauer 2009).

When the sampling plans at the left panel are entered, the right panel visualizes the plan by an OC-curve or a plot of the ASN as a function of the lot fraction nonconforming. As well the acceptable quality level p_{AQL} as the rejectable quality level p_{RQL} are indicated on the OC-curve by vertical lines. The acceptance probability at p_{AQL} should be approximately $1 - \alpha$; the acceptance probability at p_{RQL} should be approximately the customer's risk β. The ASN curves are calculated using formulas [6] and [10]. For the SSP plans, the ASN curve will be given by a horizontal line. For a DSP plan, the lots will usually be accepted in the first stage when the quality is very good and rejected in the first stage when the quality is very bad. When lots are of intermediate quality, a second sample will be required and the total number of items that need to be inspected will increase.

Case studies

We discuss three case studies at three different companies that are active in the food industry: (i) an apple juice company, (ii) a cheese company, and (iii) a company that processes eggs.

Case study I An apple juice company inspects a lot of 150,000 apples. Each lot is conforming or nonconforming towards the amount of rotten apples (an apple is rotten when patulin is present). The company uses an SSP plan by attributes with sample size $n = 50$ and an acceptance number $c = 2$ and is interested whether a DSP plan is more appealing concerning average inspection effort.

The company has agreed with local farmers to set the acceptable quality level p_{AQL} to 1% and the rejectable quality level to $p_{RQL} = 9\%$. The company is interested in an answer on the following questions:

(i) What is the supplier's and customer's risk of the SSP-(50, 2) plan?
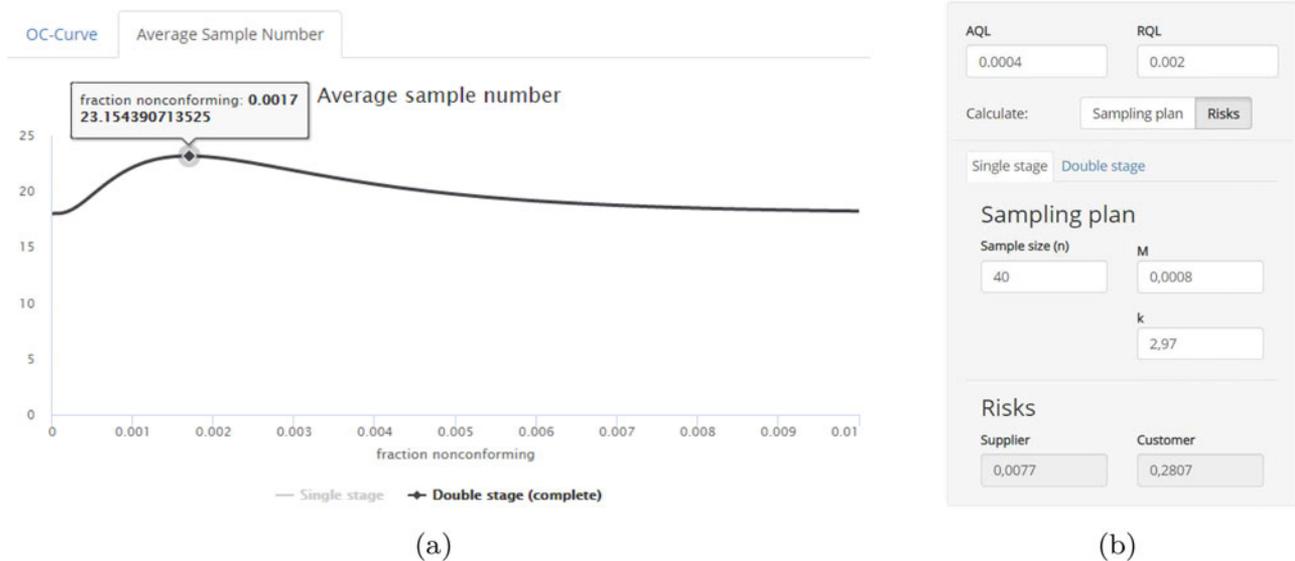
Figure 6. (a) The ASN curve of a DSP-$(18, 18, 2.85, 3.02)$ for variables inspection. (b) The risks associated to an individual ISO 3951-1 SSP-$(40, 2.97)$ plan for inspection by variables.

(ii) Which two-point SSP plan corresponds to a supplier's and customer's risk of 5 and 10%, respectively?

(iii) Which corresponding two-point DSP plan can be used and will this plan lead to a decrease of the ASN at an operating proportion nonconforming of $p = p_{AQL} = 1\%$?

To answer question 1, the left panel of the attributes inspection sheet can be used. The supplier's risk and customer's risks of the operating attributes SSP-(50, 2) plan are returned as 1.38 and 16.05%, respectively after entering the sampling plan parameters. Switching to the calculation of an SSP (Figure 4), the web tool advices an SSP-(58, 2) answering question 2. For a DSP plan, sample sizes $n_1 = n_2 = 32$ and criteria $(c_1, c_2) = (0, 2)$ are advised (Figure 5(a)). An ASN of approximately 41 is returned by reading the value of the ASN curve (complete inspection) corresponding to $p = p_{AQL}$.

Case study II An outgoing shipment of 2500 pieces of cheese is inspected. The pH of such a shipment should be at most $U = 7.00$. The quality levels p_{AQL} and p_{RQL} are set to 0.06% (6 out of 10,000) and 0.5%, respectively. The company is interested in the following questions:

(i) Which two-point SSP and DSP plan is appropriate to limit producer's and customer's risk to 5%?

(ii) What is the maximum sampling effort that can be expected when the DSP plan is used?

Figure 5(b) and 5(c) shows how to obtain the desired SSP and DSP plan, respectively. For an SSP plan, a sample size of $n = 25$ and an acceptance constant $k = 2.91$ is advised. Equivalently, an estimation of the lot fraction nonconforming should be below $M = 0.07\%$ in order to accept the lot. For a DSP plan, the sample sizes are given by $n_1 = n_2 = 18$ and the criteria in the first and second stage are given by $k_1 = 2.85$ and $k_2 = 3.02$, respectively. The corresponding ASN curve shows a maximum sample size of approximately 23 at $p = 0.17\%$ which gives an indication of the maximum sampling effort that can be expected (Figure 6(a)).

Case study III A food company that is specialized in egg processing daily receives multiple shipments of eggs that are being cooked and peeled. The freshness of the eggs is essential to minimize drop-out during peeling. For this purpose, the Haugh unit is measured, which is a measure of egg protein quality that is based on the height of the thick albumen (egg white) that immediately surrounds the yolk. Testing is destructive as the Haugh unit can only be measured by breaking the egg. The frequency of shipments is 4 per day where each lot consists of 324,000 eggs. The company and supplier who delivers the eggs agree to set the quality levels p_{AQL} and p_{RQL} as 0.04 and 0.2%, respectively. The supplier's and customer's risk are set to 5 and 10%, respectively. A lower specification limit of $L = 65$ is set to the measured Haugh unit.

The use of the tables of the ISO 3951-1 results in an SSP-$(40, 2.97)$ (standard inspection level II, σ-method). Associated supplier's and customer's risk are given by 0.77 and 28.07%, respectively (Figure 6(b)). The focal

point of the ISO 3951-1 standard is the acceptable quality level which ranges from 0.01% to 10%. To control the customer's risk up to a certain level, three general levels are available. The use of level III results in a sample size of $n = 50$ and an acceptance constant of $k = 3.01$ and reduces the customer's risk to 17.56%. However, by use of the web tool one can achieve a two-point SSP-$(39, 3.09)$ plan corresponding to a supplier's and customer's risk not exceeding 5 and 10%, respectively. Alternatively, a matching two-point DSP-$(28, 28, 3.04, 3.18)$ can be used.

Conclusion

In this article, we introduced novel search routines for the design of DSP plans for inspection by attributes and variables. Analytic properties of DSP plans were derived that were given a mathematical proof. We found that several constraints hold on the sample sizes and the criteria of two-point attributes DSP plans leading to a more efficient search routine compared to existing ones. For variables DSP plans, a routine was developed that lead to more accurate sample sizes when compared to existing tables and routines.

Furthermore, we introduced and discussed a user-friendly and interactive web tool to design and analyze SSP and DSP plans and that is freely available at www.acceptancesampling.com. In comparison with existing solutions the tool is an interactive applet that supports a two-point design of DSP plans as well as SSP plans. The use of the web tool was demonstrated on several case studies. In comparison with international standards the tool can be used to obtain a statistical interpretation of the sampling plans in terms of the consumer's and the producer's risks.

Several interesting future research directions are possible. First, extensions of our search routines to the design of sampling plans with more than two stages can be studied. Second, the design of variables DSP plans can be extended to the case of unknown variance or double specification limits. Third, algorithms can be developed that are independent of constraints on the sample sizes n_1 and n_2.

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References

Adams, R., and C. Essex. 2009. *Calculus: A complete course*. 7th ed. Canada: Pearson Education.

Balamurali, S., and C.-H. Jun. 2007. Multiple dependent state sampling plans for lot acceptance based on measurement data. *European Journal of Operational Research* 180 (3):1221–30. doi:10.1016/j.ejor.2006.05.025.

Balamurali, S., and M. Usha. 2013. Optimal designing of variables chain sampling plan by minimizing the average sample number. *International Journal of Manufacturing Engineering* 2013:1–12. doi:10.1155/2013/751807.

Bertoni, C. 2016. Sample simplification. *Quality Progress* 8: 23–7.

Cheng, T.-M., and Y.-L. Chen. 2007. A GA mechanism for optimizing the design of attribute double sampling plan. *Automation in Construction* 16 (3):345–53. doi:10.1016/j.autcon.2006.07.003.

Chow, B., P. Dickinson, and H. Hughes. 1972. A computer program for the solution of double sampling plans. *Journal of Quality Technology* 4 (4):205–9. doi:10.1080/00224065.1972.11980552.

Collani, E. V., and R. Göb. 2008. Acceptance sampling in modern industrial environments. In *Encyclopedia of statistics in quality and reliability*, eds. F. Ruggeri, R. Kenett, and F. Falting. Hoboken, NJ: John Wiley & Sons.

Dodge, H., and H. Romig. 1941. Single sampling and double inspection tables. *Bell System Technical Journal* 20 (1):1–61. doi:10.1002/j.1538-7305.1941.tb00851.x.

Duarte, B., and P. Saraiva. 2013. An optimization-based framework for designing acceptance sampling plans by

variables for non-conforming proportions. *International Journal of Quality & Reliability Management* 27 (7): 794–814. doi:10.1108/02656711011062390.

Duckett, J. 2014. *Web design with HTML, CSS, JavaScript and jQuery set*. IN: Wiley.

Hahn, G. 1974. Minimum size sampling plans. *Journal of Quality Technology* 6 (3):121–7. doi:10.1080/00224065.1974.11980633.

Hailey, W. 1980. Minimum sample size single sampling plans: A computerized approach. *Journal of Quality Technology* 12 (4):230–5. doi:10.1080/00224065.1980.11980970.

Jennett, W., and B. Welch. 1939. The control of proportion defective as judged by a single quality characteristic varying on a continuous scale. *Supplement to the Journal of the Royal Statistical Society* 6 (1):80–8. doi:10.2307/2983626.

Kiermeier, A. 2008. Visualizing and assessing acceptance sampling plans: The R package acceptance sampling. *Journal of Statistical Software* 26 (6):1–20. doi:10.18637/jss.v026.i06.

Krumbholz, W., and A. Rohr. 2009. Double ASN minimax sampling plans by variables when the standard deviation is unknown. *Asta Advances in Statistical Analysis* 93 (3): 281–94. doi:10.1007/s10182-009-0111-8.

Luca, S. 2018. Modified chain sampling plans for lot inspection by variables and attributes. *Journal of Applied Statistics* 45 (8):1447–64. doi:10.1080/02664763.2017.1375084.

Minitab, I. 2018. Minitab 18 statistical software. www.minitab.com.

Montgomery, D. C. 2013. *Introduction to statistical quality control*. 7th ed. USA: John Wiley & Sons.

Neubauer, D., and S. Luko. 2012. Comparing acceptance sampling standards, part 1. *Quality Engineering* 25 (1): 73–7. doi:10.1080/08982112.2013.738146.

Neubauer, D., and S. Luko. 2013. Comparing acceptance sampling standards, part 2. *Quality Engineering* 25 (2): 181–7. doi:10.1080/08982112.2013.758557.

Newman, R., and S. Yu. 2018. An alternative approach to accept on zero and accept on one sampling plans. *Quality Engineering* 30 (2):183–94. doi:10.1080/08982112.2017.1328064.

Olorunniwo, F., and J. Salas. 1982. An algorithm for determining double attribute sampling plans. *Journal of Quality Technology* 14 (3):166–71. doi:10.1080/00224065.1982.11978810.

R Core Team. 2018. *R: A language and environment for statistical computing*. Vienna, Austria: R Foundation for Statistical Computing.

Santos Fernãndez, E. 2016. *Acceptance sampling for food quality assurance*. Ph.D. thesis, Massey University.

Schilling, E. G., and D. V. Neubauer. 2009. *Acceptance sampling in quality control*. 2nd ed. *Statistics: A series of textbooks and monographs*. Boca Raton, FL: CRC Press.

Schmueli, G. 2016. *Practical acceptance sampling: A hands-on guide*. 2nd ed. USA: Axelrod Schnall Publishers.

Sommers, D. 1981. Two-point double variables sampling plans. *Journal of Quality Technology* 13 (1):25–30. doi:10.1080/00224065.1981.11980982.

Spurlock, J. 2013. *Bootstrap*. USA: O'Reilly Media, Inc.

Taylor, W. 1997. Selecting statistically valid sampling plans. *Quality Engineering* 10 (2):365–70. doi:10.1080/08982119708919144.

Vangjeli, E. 2012. ASN-minimax double sampling plans by variables for two-sided specification limits when the standard deviation is known. *Statistical Papers* 53 (1): 229–38. doi:10.1007/s00362-010-0331-8.

Vijayaraghavan, R. 2007. Minimum size double sampling plans for large isolated lots. *Journal of Applied Statistics* 34 (7):799–806. doi:10.1080/02664760701240287.

White, K. P. J., and K. Johnson. 2013. A framework for the derivation and verification of variables acceptance sampling plans. *Revista Investigación Operacional* 34 (3): 220–9.

White, K. P., J. Johnson, K. J. Creasey. and R. R. 2009. Attribute acceptance sampling as a tool for verifying requirements using Monte Carlo simulation. *Quality Engineering* 21 (2):203–14. doi:10.1080/08982110902723511.

Zelinka, I. 2015. A survey on evolutionary algorithms dynamics and its complexity – Mutual relations, past, present and future. *Swarm and Evolutionary Computation* 25:2–14. doi:10.1016/j.swevo.2015.06.002.

Appendix A: Proofs

Proof of Property 1

(i) Due to Eq. [5]:

$$\phi_a^{dsp}(p_{\text{RQL}}, n_1, n_2, c_1 + 1, c_2) - \phi_a^{dsp}(p_{\text{RQL}}, n_1, n_2, c_1, c_2)$$
$$= P(D_1 = c_1 + 1) - P(D_1 = c_1 + 1)P(D_2 \leq c_2 - c_1 - 1) > 0$$

and

$$\phi_a^{dsp}(p_{\text{RQL}}, n_1, n_2, c_1, c_2 + 1) - \phi_a^{dsp}(p_{\text{RQL}}, n_1, n_2, c_1, c_2)$$
$$= \sum_{j=c_1+1}^{c_2+1} P(D_1 = j)P(D_2 \leq c_2 + 1 - j)$$
$$- \sum_{j=c_1+1}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j)$$
$$= \sum_{j=c_1+1}^{c_2} P(D_1 = j)(P(D_2 \leq c_2 - j + 1) - P(D_2 \leq c_2 - j))$$
$$+ P(D_1 = c_2 + 1)P(D_2 = 0) > 0$$

showing that $\phi_a^{dsp}(p, n_1, n_2, c_1, c_2)$ is strictly increasing as a function of c_1 and c_2.

(ii) Due to the properties of the binomial distribution the probabilities $P(D_1 \leq c_1)$ and $P(D_1 = j)$ in [5] decrease as a function of n_1. Furthermore, the probabilities $P(D_2 \leq c_2 - j)$ decrease as a function of n_2.

(iii) From the operating procedure of a DSP plan, we know that when the number of nonconforming items D_1 in the first sample is situated between c_1 and c_2, the lot sentencing depends on a second sample taken from the lot. Clearly, for an SSP-(n_1, c_2), the lot is always accepted for $c_1 < D_1 \leq c_2$ such that the acceptance probability of the plan will increase with respect to a
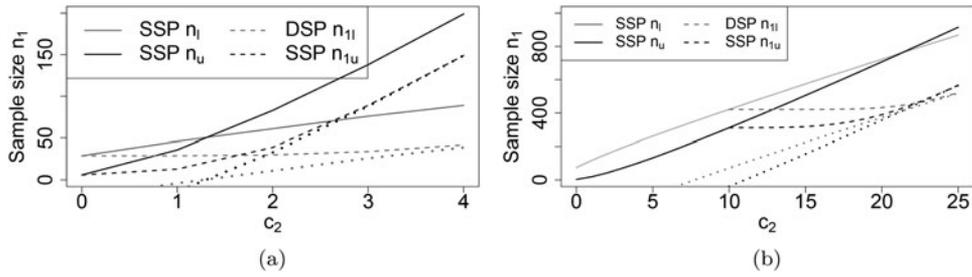
Figure A1. Examples of curves $n_l(c_2)$ and $n_u(c_2)$ as defined in [11] for an SSP-(n, c_2) plan and $n_{1l}(n_2, c_1, c_2)$ and $n_u(n_2, c_1, c_2)$ as defined in [A.1] for a DSP-(n_1, n_2, c_1, c_2) plan with fixed choices of c_1 and n_2: (a) $c_1 = 0$ and $n_2 = 50$, with $p_{AQL} = 1\%$ and $p_{RQL} = 10\%$, (b) $c_1 = 10$ and $n_2 = 350$, with $p_{AQL} = 2\%$ and $p_{RQL} = 4\%$. Risks are set to $\alpha = \beta = 5\%$. The gray and black dotted lines correspond to the curves $n_l(c_2) - n_2$ and $n_u(c_2) - n_2$ respectively to which $n_{1l}(n_2, c_1, c_2)$ and $n_u(n_2, c_1, c_2)$ converge for large c_2.

DSP-(n_1, n_2, c_1, c_2). For an SSP-(n_1, c_1), the lot is always rejected for $c_1 < D_1 \leq c_2$ such that the acceptance probability will decrease. When $c_1 = c_2$, the condition to enter the second stage of a DSP cannot be satisfied anymore resulting in an SSP-(n_1, c_1) procedure. The summation in [5] disappears and simplifies to $P(D_1 \leq c_1)$ which is the acceptance probability of the SSP-(n_1, c_1) plan.

(iv) First, note that, due to Property (ii), we can find (analogously as in [11]) a lower - and an upper bound on the sample size n_1 that depend on the other parameters n_2, c_1, c_2:

$$n_{1l}(n_2, c_1, c_2) = \inf\left\{n_1 | \phi_a^{dsp}(p_{RQL}, n_1, n_2, c_1, c_2) \leq \beta\right\} \quad \text{and}$$

$$n_{1u}(n_2, c_1, c_2) = \sup\left\{n_1 | \phi_a^{dsp}(p_{AQL}, n_1, n_2, c_1, c_2) \geq 1 - \alpha\right\}.$$
[A.1]

To prove the constraint $n_1 + n_2 \geq n_l(c_2)$, we consider a two-point DSP-(n_1, n_2, c_1, c_2) and verify that:

$$\beta \geq \phi_a^{dsp}(p_{RQL}, n_1, n_2, c_1, c_2)$$

$$= P(D_1 \leq c_1) + \sum_{j=c_1+1}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j)$$

$$= \sum_{j=0}^{c_1} P(D_1 = j) + \sum_{j=c_1+1}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j)$$

$$\geq \sum_{j=0}^{c_1} P(D_1 = j)P(D_2 \leq c_2 - j) + \sum_{j=c_1+1}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j)$$

$$= \sum_{j=0}^{c_2} P(D_1 = j)P(D_2 \leq c_2 - j)$$

$$= P(D_1 + D_2 \leq c_2)$$

$$= \phi_a^{ssp}(p_{RQL}, n_1 + n_2, c_2),$$

such that $n_1 + n_2 \geq n_l(c_2)$ by the definition of $n_l(c_2)$ given in [11]. Note that, for $j \leq c_1, P(D_2 \leq c_2 - j) \rightarrow 1$ as $c_2 \rightarrow n_2 + j$. Therefore, for fixed n_2, $n_{1l}(n_2, c_1, c_2) + n_2$ tends to $n_l(c_2)$ as c_2 tends to $n_2 + c_1$. Similarly, by replacing p_{RQL} by p_{AQL} in the above inequality, one can show that $n_{1u}(n_2, c_1, c_2) + n_2$ tends to $n_u(c_2)$ as c_2 tends to $n_2 + c_1$.

We proceed by proving that $c_2 \geq c^*$. For $c_1 \geq c^*$, this is obvious as $c_1 \leq c_2$. Consider some fixed first sample acceptance number $c_1 < c^*$ and let $c_2 \geq c_1$ and $n_2 \geq 0$. Clearly, if $c_2 = c_1$, then $n_l(c_1) = n_{1l}(n_2, c_1, c_2)$ and $n_u(c_1) = n_{1u}(n_2, c_1, c_2)$ due to Property 1(iii). Therefore, as $c_1 < c^*, n_{1l}(n_2, c_1, c_1) > n_{1u}(n_2, c_1, c_1)$. For large c_2, $n_{1l}(n_2, c_1, c_2) + n_2$ tends to $n_l(c_2)$ and $n_{1u}(n_2, c_1, c_2) + n_2$

tends to $n_u(c_2)$ such that $n_{1l}(n_2, c_1, c_2) < n_{1u}(n_2, c_1, c_2)$ for large c_2. Denote $\bar{c}$ as the minimum number c_2 such that $n_{1l}(n_2, c_1, c_2) < n_{1u}(n_2, c_1, c_2)$. For $n_2 = 0$, a two-point DSP-(n_1, n_2, c_1, c_2) is equivalent with an SSP-(n_1, c_2) plan for which $n_l(c_2) < n_u(c_2)$ when $c_2 \geq c^*$. Thus, for $n_2 = 0$, the minimum acceptance number $\bar{c}$ is given by c^*. For choices $n_2 > 0$, the minimum value $\bar{c}$ for c_2 cannot be smaller than c^* in order to maintain a producer's risk of at most α (Property 1(i)). Therefore, $\bar{c} \geq c^*$. In Figure A1, the intersection of the curves $n_{1l}(n_2, c_1, c_2)$ and $n_{1u}(n_2, c_1, c_2)$ is illustrated for fixed choices of c_1 and n_2.

For any DSP plan with $c_1 = c^*$ and $c_2 \geq c^*$ a sample size $n_1 \geq n^*$ is required to keep the consumer's risk below β. Indeed, the consumer's risk of any DSP-(n^*, n_2, c^*, c_2) plan will be higher than that of an SSP-(n^*, c^*) plan as (due to (i)):

$$\phi_a^{dsp}(p_{RQL}, n^*, n_2, c^*, c_2) \geq \phi_a^{dsp}(p_{RQL}, n^*, n_2, c^*, c^*)$$
$$= \phi_a^{ssp}(p_{RQL}, n^*, c^*) \approx \beta.$$
[A.2]

Therefore, due to (ii), a consumer's risk lower than β can only be obtained for sample sizes $n_1 \geq n^*$ when $c_1 = c^*$. Thus, as c_1 approaches c^*, a minimum sample size n_1 is achieved by the SSP-(n^*, c^*) plan. Finally, a two-point DSP with an OC-curve passing through $(p_{AQL}, 1 - \alpha)$ and (p_{RQL}, β) obviously exists for $n_1 \leq n^*$ and $c_1 \leq c^* \leq c_2$ as the SSP-(n^*, c^*) is one such plan with $n_2 = 0$ and $c_1 = c_2 = c^*$. As n_2 increases, the consumer's risk decreases and the minimum sample size n_1 won't exceed n^*.

Proof of Property 2

In the following lemma, we derive some important integrals of the density function of the bivariate normal distribution that will be required to prove Property 2.

Lemma A.1. *Consider the density function $f(u, v)$ of the bivariate normal distribution $N(0, \Sigma)$ with covariance Σ as in [8]:*

$$f(u, v) = \frac{1}{2\pi}\sqrt{\frac{N}{n_2}}\exp\left[-\frac{N}{2n_2}\left(u^2 - 2\sqrt{\frac{n_1}{N}}uv + v^2\right)\right], \quad [A.3]$$

with $N = n_1 + n_2$. For each $u_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}$, the following identities apply:

$$\int_{-\infty}^a f(u, v_0)du = \frac{1}{\sqrt{2\pi}}P\left(Z \leq \sqrt{\frac{N}{n_2}}\left[a - \sqrt{\frac{n_1}{N}}v_0\right]\right)e^{-\frac{v_0^2}{2}} \quad [A.4]$$

and:

$$\int_{-\infty}^{a} f(u_0, v)dv = \frac{1}{\sqrt{2\pi}} P\left(Z \le \sqrt{\frac{N}{n_2}}\left[a - \sqrt{\frac{n_1}{N}}u_0\right]\right)e^{-\frac{u_0^2}{2}} \quad [A.5]$$

Proof. The expression of $f(u, v)$ can be integrated with respect to the variable v and using a substitution $w = \sqrt{\frac{N}{n_2}}(u - \sqrt{\frac{n_1}{N}}v_0)$:

$$\int_{-\infty}^{a} f(u, v_0)du = \frac{1}{2\pi}\sqrt{\frac{N}{n_2}}\left(\int_{-\infty}^{a}\right.$$

$$\exp\left[-\frac{N}{2n_2}\left(u^2 - 2\sqrt{\frac{n_1}{N}}uv_0\right)\right]du\right)e^{-\frac{N}{2n_2}v_0^2}$$

$$= \frac{1}{2\pi}\sqrt{\frac{N}{n_2}}\left(\int_{-\infty}^{a}\right.$$

$$\exp\left[-\frac{N}{2n_2}\left(u - \sqrt{\frac{n_1}{N}}v_0\right)^2\right]du\right)e^{\frac{n_1}{2n_2}v_0^2}e^{-\frac{N}{2n_2}v_0^2}$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{\sqrt{\frac{N}{n_2}}\left(a - \sqrt{\frac{n_1}{N}}v_0\right)} e^{-\frac{w^2}{2}}dw\right)e^{-\frac{v_0^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} P\left(Z \le \sqrt{\frac{N}{n_2}}\left[a - \sqrt{\frac{n_1}{N}}u_0\right]\right)e^{-\frac{v_0^2}{2}}$$

The identity [A.5] immediately follows by applying the symmetry property of $f(u, v)$ about the line $v = u$, that is, $f(u, v) = f(v, u)$. □

We now proceed by proving Property 2:

i. Clearly when the acceptance constant k_1 increases the size of the probability event in [9] reduces implying a decrease in the expression of the acceptance probability ϕ_v^{dsp} in [7]. Moreover, for criteria $k_2 < k_2'$, and denoting $u = \sqrt{n_1}(Z_p - k_2)$, $u' = \sqrt{n_1}(Z_p - k_2')$, and $v = \sqrt{n_1 + n_2}(Z_p - k_1)$, one obtains:

$$\phi_v^{dsp}(p, n_1, n_2, k_1, k_2') - \phi_v^{dsp}(p, n_1, n_2, k_1, k_2)$$
$$= -P(u' \le Z \le u) + P(W_1 \le v, u' \le W_2 \le u)$$
$$< -P(u' \le Z \le u) + P(u' \le W_2 \le u)$$
$$= -P(u' \le Z \le u) + P(u' \le Z \le u) = 0$$

as the marginal distribution of W_2 is a standard normal distribution $N(0, 1)$.

ii. To study the dependency on n_1 and n_2, we consider the expression of the OC-curve defined by Eqs. [7] and [9]

$$\phi_v^{dsp}(p, n_1, n_2, k_1, k_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{n_1}(Z_p - k_2)} e^{-\frac{u^2}{2}}du$$
$$+ \int_{-\infty}^{\sqrt{N}(Z_p - k_1)}\int_{\sqrt{n_1}(Z_p - k_2)}^{\sqrt{n_1}(Z_p - k_1)} f(u, v)dudv \quad [A.6]$$

with $f(u, v)$, the density function of the bivariate normal distribution $N(\mathbf{0}, \Sigma)$ as defined in [A.3] and $N = n_1 + n_2$. The boundaries as well as the integrand $f(u, v)$ in [A.6] depend on the sample sizes n_1 and n_2 leading to complex expressions of the derivatives with respect to n_1 and n_2. Instead, we will study the slopes:

$$\frac{\partial\phi_v^{dsp}}{\partial p}(p, n_1, n_2, k_1, k_2),$$

using Lemma A.1 and we will show that these are increasing with respect to n_1 and n_2 for proportions nonconforming near zero and near 1. In particular for $n_1 < n_1'$ and some k_1, k_2, n_2, we will show that there exists some $p_0 > 0$ (resp. $p_1 < 1$) such that for p in $]0, p_0[$ (resp. for p in $]p_1, 1[$):

$$\frac{\partial\phi_v^{dsp}}{\partial p}(p, n_1, n_2, k_1, k_2) < \frac{\partial\phi_v^{dsp}}{\partial p}(p, n_1', n_2, k_1, k_2) \quad [A.7]$$

and therefore by integration over $[0, p]$ (resp. $[p, 1]$), one obtains for p in $]0, p_0[$ (resp. for p in $]p_1, 1[$):

$$\phi_v^{dsp}(p, n_1, n_2, k_1, k_2) < \phi_v^{dsp}(p, n_1', n_2, k_1, k_2)$$
$$\left(resp. \phi_v^{dsp}(p, n_1, n_2, k_1, k_2) > \phi_v^{dsp}(p, n_1', n_2, k_1, k_2)\right). \quad [A.8]$$

Moreover, due to the continuity of the OC-curves of DSP plans, the OC-curves of the DSP-(n_1, n_2, k_1, k_2) and DSP-(n_1', n_2, k_1, k_2) will intersect somewhere in $]0, 1[$. A similar reasoning can be used to study the dependency on n_2.

We proceed by proving the inequalities in [A.7]. The slope of the first term is given by:

$$\frac{\partial}{\partial p}\left(P(Z \le \sqrt{n_1}(Z_p - k_2))\right) = -\sqrt{n_1}e^{-\frac{n_1}{2}(Z_p - k_2)^2}e^{\frac{Z_p^2}{2}}, \quad [A.9]$$

where we used (inverse function theorem (Adams and Essex 2009)):

$$\frac{\partial Z_p}{\partial p} = -\sqrt{2\pi}e^{\frac{Z_p^2}{2}}. \quad [A.10]$$

To study the dependency on n_1 of this slope, we calculate the following second order partial derivative using [A.9]:

$$\frac{\partial^2}{\partial n_1 \partial p}\left(P(Z \le \sqrt{n_1}(Z_p - k_2))\right)$$
$$= \left(-\frac{1}{2\sqrt{n_1}} + \frac{\sqrt{n_1}}{2}(Z_p - k_2)^2\right)e^{-\frac{n_1}{2}(Z_p - k_2)^2}e^{\frac{Z_p^2}{2}}, \quad [A.11]$$

which is clearly positive when $\frac{\sqrt{n_1}}{2}(Z_p - k_2)^2 > \frac{1}{2\sqrt{n_1}}$, which holds for p close to zero (resp. close to 1) because $(Z_p - k_2)^2 \to +\infty$ as $p \to 0$ (resp. $p \to 1$). Note that the definition of the acceptance probability $\phi_v^{dsp}(p, n_1, n_2, k_1, k_2)$ can be extended to allow positive real numbers n_1 and n_2 such that partial derivatives to n_1 and n_2 are well-defined.

Taking the derivative of the second term in [A.6], which we denote as P_2, leads to:

$$\frac{\partial P_2}{\partial p} = \frac{\partial}{\partial p}\left(\int_{-\infty}^{\sqrt{N}(Z_p - k_1)}\int_{\sqrt{n_1}(Z_p - k_2)}^{\sqrt{n_1}(Z_p - k_1)} f(u, v)dudv\right)$$
$$= \left[\left(\int_{-\infty}^{\sqrt{N}(Z_p - k_1)} (p(\sqrt{n_1}(Z_p - k_1), v)\right.\right.$$
$$\left. - p(\sqrt{n_1}(Z_p - k_2), v))dv\right)\sqrt{n_1}$$
$$\left. + \left(\int_{\sqrt{n_1}(Z_p - k_2)}^{\sqrt{n_1}(Z_p - k_1)} p(u, \sqrt{N}(Z_p - k_1))du\right)\sqrt{N}\right]\frac{\partial Z_p}{\partial p}$$
$$= f_0 - g_0 + h_0,$$

where, based on Lemma A.1, we defined:

$$f_0 = \frac{1}{\sqrt{2\pi}} P\big(Z \leq \sqrt{n_2}(Z_p - k_1)\big) \sqrt{n_1} e^{-\frac{n_1}{2}(Z_p - k_1)^2} \frac{\partial Z_p}{\partial p},$$

$$g_0 = \frac{1}{\sqrt{2\pi}} P\Big(Z \leq \sqrt{n_2}(Z_p - k_1) + \frac{n_1}{\sqrt{n_2}}(k_2 - k_1)\Big)$$
$$\sqrt{n_1} e^{-\frac{n_1}{2}(Z_p - k_2)^2} \frac{\partial Z_p}{\partial p},$$

$$h_0 = \frac{1}{\sqrt{2\pi}} P\Big(-\sqrt{\frac{N}{n_2}}\sqrt{n_1}(k_2 - k_1) \leq Z \leq 0\Big)$$
$$\sqrt{N} e^{-\frac{N}{2}(Z_p - k_1)^2} \frac{\partial Z_p}{\partial p}.$$

The functions f_0, g_0 and h_0 increase as a function of n_1 for p near zero. Indeed, the dependency of the exponential factors are similar to [A.9] while the probabilistic factors clearly increase as a function of n_1. We now obtain:

$$\frac{\partial^2 \phi_v^{dsp}}{\partial n_1 \partial p} = \frac{\partial f_0}{\partial n_1} + \frac{\partial h_0}{\partial n_1}$$
$$+ \Big[\frac{\partial^2}{\partial n_1 \partial p}\big(P(Z \leq \sqrt{n_1}(Z_p - k_2))\big) - \frac{\partial g_0}{\partial n_1}\Big], \quad [\text{A.12}]$$

where $\frac{\partial f_0}{\partial n_1}$ and $\frac{\partial h_0}{\partial n_1}$ are positive for p near zero. Furthermore, by using [A.11] and [A.10], we find:

$$\frac{\partial^2}{\partial n_1 \partial p}\big(P(Z \leq \sqrt{n_1}(Z_p - k_2))\big) - \frac{\partial g_0}{\partial n_1}$$
$$= \Big(-\frac{1 - \beta_p}{2\sqrt{n_1}} + (1 - \beta_p)\frac{\sqrt{n_1}}{2}(Z_p - k_2)^2 + \sqrt{n_1}\frac{\partial \beta_p}{\partial n_1}\Big)$$
$$e^{-\frac{n_1}{2}(Z_p - k_2)^2} e^{\frac{Z_p^2}{2}} \quad [\text{A.13}]$$

where we defined the probability:

$$\beta_p = P\Big(Z \leq \sqrt{n_2}(Z_p - k_1) + \frac{n_1}{\sqrt{n_2}}(k_2 - k_1)\Big).$$

Clearly β_p increases as a function of n_1 such that $\frac{\partial \beta_p}{\partial n_1} \geq 0$. For p near zero, one can suppose that $\frac{\sqrt{n_1}}{2}(Z_p - k_2)^2 > \frac{1}{2\sqrt{n_1}}$ implying that the expression in [A.13] is positive. From [A.12], we conclude that $\frac{\partial^2 \phi_v^{dsp}}{\partial n_1 \partial p} \geq 0$ for p near zero. Also, proportions noncon-

forming p near 1 lead to positive expressions in [A.13] and [A.12] such that the inequalities in [A.7] hold. A similar reasoning can be used to prove that the slopes $\frac{\partial \phi_v^{dsp}}{\partial p}$ are increasing as a function of n_2 for p near 0 (resp. near 1).

(iii) The proof proceeds completely similar to the proof of Property 1(iii).
(iv) Following the same lines as in the proof of Property 1(iv), one obtains for $k_1 < k^*$ (due to (ii), assuming $p_{\text{RQL}} > p_1$):

$$\phi_v^{dsp}(p_{\text{RQL}}, n^*, n_2, k_1, k^*) > \phi_v^{dsp}(p_{\text{RQL}}, n^*, n_2, k^*, k^*)$$
$$= \phi_v^{ssp}(p_{\text{RQL}}, n^*, k^*) \approx \beta.$$

Only a sample size n_1 of at least n^* can result in a consumer's risk below β, when $k_2 = k^*$. Therefore, as $k_2 \downarrow k^*$, a minimum sample size n_1 is achieved by the SSP-(n^*, k^*) plan. Furthermore, a two-point DSP with an OC-curve passing through $(p_{\text{AQL}}, 1 - \alpha)$ and (p_{RQL}, β) obviously exists for $n_1 \leq n^*$ as the SSP-(n^*, k^*) is one such plan with $n_2 = 0$ and $k_1 = k_2 = k^*$. As n_2 increases, the consumer's risk decreases and the minimum sample size n_1 won't exceed n^*.

To prove that $k_1 \leq k^* \leq k_2$, we show that both $k_1 \leq k_2 \leq k^*$ and $k^* \leq k_1 \leq k_2$ are not possible. Indeed, for $k_1 \leq k_2 \leq k^*$ and $n_1 \leq n^*$, one finds (due to (i) and (ii), assuming $p_{\text{RQL}} > p_1$):

$$\phi_v^{dsp}(p_{\text{RQL}}, n_1, n_2, k_1, k_2) \geq \phi_v^{dsp}(p_{\text{RQL}}, n_1, n_2, k^*, k^*)$$
$$= \phi_a^{ssp}(p_{\text{RQL}}, n_1, k^*) \geq \beta$$

such that the consumer's risk exceeds β. Similar, for $k^* \leq k_1 \leq k_2$ and $n_1 \leq n^*$, one finds (due to (i), (ii) and (iii) and assuming $p_{\text{AQL}} < p_0$):

$$\phi_v^{dsp}(p_{\text{AQL}}, n_1, n_2, k_1, k_2)$$
$$\leq \phi_v^{dsp}(p_{\text{AQL}}, n^*, n_2, k_1, k_2) \leq \phi_a^{ssp}(p_{\text{AQL}}, n^*, k_1)$$
$$\leq \phi_a^{ssp}(p_{\text{AQL}}, n^*, k^*)$$
$$\leq 1 - \alpha$$

such that the producer's risk exceeds α.

Appendix B: Table of matching single and double sampling plans

Table B1. Table of two-point SSP and DSP plans indexed by p_{AQL} and p_{RQL}.

		SSP-(n, c)		DSP-(n_1, n_2, c_1, c_2)				SSP-(n, k)		DSP-(n_1, n_2, k_1, k_2)			
p_{AQL}	p_{RQL}	n	c	n_1	c_1	c_2	ASN	n	k	n_1	k_1	k_2	ASN
0.001	0.003	3922	7	2391	3	8	2913.4	74	2.90	54	2.86	2.96	60.8
	0.004	2317	5	1384	2	5	1604.8	45	2.85	33	2.80	2.93	37.1
	0.005	1335	3	692	0	3	1033.9	33	2.80	24	2.75	2.90	26.9
	0.006	1112	3	681	1	3	779	26	2.77	19	2.71	2.87	21.3
	0.007	759	2	416	0	2	554	22	2.74	16	2.67	2.84	17.8
0.002	0.006	1960	7	1195	3	8	1455.8	64	2.67	47	2.63	2.74	53.2
	0.008	1158	5	692	2	5	802.4	39	2.61	29	2.57	2.70	32.4
	0.01	667	3	346	0	3	517	29	2.57	21	2.51	2.66	23.4
	0.012	555	3	340	1	3	388.8	23	2.54	17	2.47	2.62	18.6
	0.014	379	2	208	0	2	277	19	2.50	14	2.43	2.6	15.4
0.005	0.015	783	7	477	3	8	580.5	53	2.35	39	2.31	2.42	43.3
	0.02	462	5	276	2	5	319.7	32	2.29	23	2.23	2.39	26.2
	0.025	266	3	138	0	3	206.2	23	2.23	17	2.17	2.33	18.8
	0.03	221	3	135	1	3	154.2	18	2.19	13	2.11	2.33	14.8
	0.035	151	2	83	0	2	110.5	15	2.15	11	2.07	2.27	12.2
0.01	0.03	390	7	238	3	8	289.3	44	2.08	32	2.03	2.16	35.9
	0.04	198	4	137	2	5	158.4	26	2.00	19	1.94	2.12	21.5
	0.05	132	3	69	0	3	103.2	19	1.95	14	1.88	2.05	15.4
	0.06	110	3	67	1	3	76.4	15	1.90	11	1.82	2.00	12.0
	0.07	75	2	41	0	2	54.5	12	1.85	9	1.77	1.98	9.90
0.02	0.04	616	18	331	7	19	445	94	1.88	69	1.85	1.94	77.6
	0.05	306	10	175	4	11	222.8	52	1.83	38	1.78	1.90	42.6
	0.06	194	7	118	3	8	142.9	35	1.78	26	1.73	1.85	28.7
	0.07	131	5	82	2	6	100.4	26	1.73	19	1.67	1.84	21.3
	0.08	98	4	68	2	5	78.4	21	1.69	15	1.62	1.81	16.9
0.03	0.05	807	32	415	10	33	706.2	154	1.75	114	1.72	1.79	128.1
	0.06	410	18	243	9	20	290.9	81	1.70	60	1.67	1.76	67.1
	0.07	252	12	130	4	12	175.7	53	1.65	39	1.61	1.72	43.5
	0.08	175	9	93	3	9	121.3	38	1.61	28	1.57	1.70	31.5
	0.09	129	7	68	2	7	90.7	30	1.58	22	1.52	1.67	24.5
0.05	0.07	1196	72	598	10	72	1196	300	1.55	222	1.53	1.58	249.5
	0.08	572	37	286	10	37	529.1	149	1.51	111	1.49	1.55	124.1
	0.09	348	24	185	10	25	244.5	93	1.47	69	1.44	1.53	77.1
	0.1	233	17	127	7	18	165.5	65	1.44	48	1.40	1.51	54.0
	0.11	170	13	94	5	14	125	49	1.41	36	1.37	1.49	40.8

Double-stage plans are chosen such that the ASN at p_{AQL} is minimized and $n_1 = n_2$. Producer's and consumer's risk are set to $\alpha = 5\%$ and $\beta = 10\%$, respectively.