Modified chain sampling plans for lot inspection by variables and attributes

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ABSTRACT

The purpose of acceptance sampling is to develop decision rules to accept or reject production lots based on sample data. When testing is destructive or expensive, dependent sampling procedures cumulate results from several preceding lots. This chaining of past lot results reduces the required size of the samples. A large part of these procedures only chain past lot results when defects are found in the current sample. However, such selective use of past lot results only achieves a limited reduction of sample sizes. In this article, a modified approach for chaining past lot results is proposed that is less selective in its use of quality history and, as a result, requires a smaller sample size than the one required for commonly used dependent sampling procedures, such as multiple dependent sampling plans and chain sampling plans of Dodge. The proposed plans are applicable for inspection by attributes and inspection by variables. Several properties of their operating characteristic-curves are derived, and search procedures are given to select such modified chain sampling plans by using the two-point method.

KEYWORDS

acceptance sampling, attributes inspection, conditional sampling, high quality processes, variables inspection

1. Introduction

One of the oldest aspects of quality assurance is concerned with inspection and decision making regarding products. In the 1930s and 1940*s* *acceptance sampling* was developed for inspection of incoming or receiving inspection [4]. Acceptance sampling describes decision rules for acceptance or rejection of a batch or lot based on the inspection and classification of a sample of units that are selected at random. It is used by industries worldwide for assuring the quality of incoming and outgoing goods. It can be used in contracting and sub-contracting, in which the contractor wants to assure the quality of the incoming goods or services, e.g., a food company that processes eggs inspects the quality of each shipment that is received from a local farmer. Acceptance sampling is also commonly used by suppliers or manufacturers themselves as a tool for audit or compliance, e.g., a well-known vendor of child safety car seats inspects the quality of its products using acceptance sampling [18]. Civilian standards as ISO standards (and their ANSI/ASQC/BS/Military Standards or other counterparts) are commonly used tools to implement acceptance sampling schemes in practice. These standards dictate the sample size to be drawn from each batch and the requirements that the sample must meet to assure that the entire batch is of acceptable quality [13].

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A single sampling (SSP) plan is a procedure where lot sentencing is based on the inspection of one random sample. Clearly, such sampling plans involve the risk that the sample will not adequately reflect the quality of the lot. The *statistical design* of a SSP plan is based on its operating characteristic (OC) curve, which shows the probability of accepting a lot given various proportions defective. The so-called two-point method requires that the OC-curve $p \mapsto \phi(p, \mathcal{P})$ of a plan $\mathcal{P}$ passes approximately through two designated points: $(p_{\mathrm{AQL}}, 1-\alpha)$ and $(p_{\mathrm{RQL}}, \beta)$. In this way, the *producer's risk* to reject a (good) lot with $p < p_{\mathrm{AQL}}$ is smaller than α and the *consumer's risk* to accept a (bad) lot with $p > p_{\mathrm{RQL}}$ is smaller than β. The shape of the OC-curve determines the discriminative power of the sampling plan. An ideal OC-curve would accept all lots with $p < p_{\mathrm{AQL}}$ and would reject all lots with $p > p_{\mathrm{AQL}}$. Ideal OC-curves can almost never be obtained in practice, such that an approximation is desired. Good approximation can be obtained by increasing the sample size, but this will not be possible when testing is costly or destructive.

An alternative solution is to rely on conditional sampling procedures which are shown to require a smaller sample size than traditional SSP plans [2, 20]. In these procedures, acceptance or rejection of a lot is based not only on the sample from that lot, but also on sample results from past or future lots. In this article, *dependent or chain sampling plans* will be studied that make use of past lot results. In *multiple dependent state sampling* (MDS) plans, past sample results are used when the quality of the current lot is doubtful. Such plans are developed for inspection by attributes [2, 20] and inspection by variables [1]. The design and construction of such plans is discussed in [2, 10]. Variables sampling plans may also incorporate capability indices to account for sampling error when estimating sample means [14, 22]. The chain sampling plans of Dodge, known as ChSP-1 plans, provide an alternative to *zero-acceptance number* plans to prevent a pathological shape of the OC-curve that is convex for all quality levels [6]. ChSP-1 plans were developed for inspection by variables as well as inspection by attributes [5, 11]. Tables and procedures to select ChSP-1 plans are studied in [3, 9, 15].

A large part of existing dependent procedures, however, only chain past lot results when defective units are found in the current lot. For instance, a classical ChSP-1 plan only chains past lot results when a defective unit is observed in the current sample. This means that the available historical evidence of quality is not fully utilized. For this reason, a modification of ChSP-1 plans was introduced in [8], termed MChSP-1 plans. These plans explore quality history even when no nonconformities are observed and result in a lower sample size than the one required for ChSP-1 plans. However the plans can only be used for inspection by attributes and their selection is only studied under the condition of a Poisson model.

This article provides extensions of MChSP-1 plans, termed *modified chain sampling* (MChSP) plans, that are applicable to both attributes and variables inspection. The attributes sampling plans will be based on the binomial model, and the variables sampling plans will use a quality characteristic following a normal distribution with known variance. Furthermore, MChSP-1 plans will be studied under the condition of a binomial model. Search procedures are developed that allow the design of such plans and several analytic properties of their OC-curves are derived. In particular, it will be shown that the OC-curve of a MChSP plan always has an inflection point. This property will ensure the discriminative power of MChSP plans [7]. Furthermore, the performance of MChSP plans will be investigated and compared with ChSP-1, MChSP-1 and MDS plans in terms of required sample sizes. As with other dependent

sampling procedures, we assume that lots are submitted substantially in the order of production with an unknown but constant proportion defective.

Because of their selective use of past lot results, most existing dependent procedures only achieve a limited reduction of sample sizes. This implies that similar sample sizes are obtained when the number i of chained results increases. In fact, minimal sample sizes for ChSP-1 and MDS plans are achieved for $1 \leq i \leq 5$ [1, 10]. A further increase in i will not result in a reduction of the sample size. The proposed MChSP plans do allow to reduce the sample size by increasing the number of chained lot results.

The remainder of the paper is structured as follows. In Section 2, an introduction to single and dependent sampling procedures is given. In Section 3 the proposed MChSP plans are introduced and several analytic properties of their OC-curves are derived. In Section 4, search procedures are provided for the selection of these plans, and a parameter sensitivity analysis is presented. Section 5 is devoted to a comparison between the performance of several dependent sampling plans. In Section 6, the use of MChSP plans is illustrated by a case study. Finally, conclusions are presented in Section 7.

2. Preliminaries

Depending upon the type of data, acceptance sampling plans can be divided into two classes. In *attributes sampling plans* measures are used that classify items as being defective or non-defective. In *variables sampling plans* quality features are measured on a numerical scale. In this section, an overview is given of the single and dependent sampling plans that will be used in this article.

The fundamental tool in the analysis and design of sampling plans is the OC-curve $p \mapsto \phi(p, \mathcal{P})$ which relates the lot fraction defective to the lot acceptance probability of the plan $\mathcal{P}$. The *two-point method* for selecting a sampling plan requires that this curve passes approximately through two designated points $(p_{\text{AQL}}, 1-\alpha)$ and (p_{RQL}, β) such that:

$$\begin{cases} \phi(p_{\text{AQL}}, \mathcal{P}) & \geq & 1-\alpha \\ \phi(p_{\text{RQL}}, \mathcal{P}) & \leq & \beta. \end{cases} \tag{1}$$

A SSP plan *by attributes* is a procedure in which decision is made to accept or reject a lot of size N based on the number of defective units of a random sample of size $n < N$ taken from it. Such a sampling plan is defined by two integers (n, c) where n denotes the sample size and c denotes the maximum number of defective units c that are allowed in order to accept the lot under inspection (the so-called *criterion*). When the lot size is large ($\frac{n}{N} < 0.1$), the number of defective units $D_{n,p}$ that is found in samples of size n drawn from the lot follows approximately a binomial distribution, i.e., $D_{n,p} \sim B(n, p)$, where p denotes the (unknown) lot fraction defective. The OC-curve $p \mapsto \phi_a(p, n, c)$ of a SSP-(n, c) plan is given by:

$$\phi_a(p, n, c) \quad = \quad P(D_{n,p} \leq c) = \sum_{j=0}^{c} \frac{n!}{j!(n-j)!}\, p^j (1-p)^{n-j},$$

and a statistical design is a solution of the system of equations (1) where $\phi = \phi_a$. Minimal sample sizes can be achieved by considering *zero-acceptance number sampling*

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plans $(n, 0)$. As noted by several authors, the OC-curve of these plans is convex for all quality levels. As a result, the producer's risk is rapidly increasing, even for small values of the lot fraction defective, which is unfair to the producer [6].

The ChSP-1 plans are an alternative on zero-acceptance number sampling plans that prevent such pathological shape of the OC-curve [13]. The operating procedure of these plans is as follows:

(1) Select a sample of size n from the current lot and observe the number of defective units $D_{n,p}$.
(2) If $D_{n,p} = 0$ the lot is accepted; If $D_{n,p} \geq 2$, the lot is rejected; And if $D_{n,p} = 1$, the lot is accepted provided there have been no defectives in the previous i lots.

The points on the OC-curve of a ChSP-1-(n, i) plan are determined by:

$$\phi_a^{\text{ChSP}}(p, n, i) = P_0(p) + P_1(p) P_0(p)^i, \tag{2}$$

where $P_c(p) = \binom{n}{c} p^c (1-p)^{n-c}$ denotes the binomial probability of having c defects. These plans only chain past results when a defective unit is observed in the current sample. Clearly, the OC-curve converges to that of a zero-acceptance sampling plan $(n, 0)$ as $i \to +\infty$.

The sample size of a ChSP-1 plan can be further reduced by exploring quality history, even when no nonconformities are observed. As a results, Govindaraju [8] introduced MChSP-1 plans that are defined by two integers: the sample size n and the number of chained sample results i. Under the condition of a bionomial model, the decision procedure of a MChSP-1-(n, i) plan is given by:

(1) Select a sample of size n from the current lot and observe the number of defective units $D_{n,p}$. Reject the lot if $D_{n,p} \geq 1$.
(2) Accept the lot if no defective units are found in the sample provided the preceding i samples are also free from defective units except in one sample that contains at most one defective unit. Otherwise reject the lot.

The OC-curve of a MChSP-1-(n, i) plan is given by:

$$\phi_a^{\text{MChSP}}(p, n, i) = P_0(p)^{i+1} + i P_0(p)^i P_1(p).$$

The acceptance probability is obtained as the sum of two events. The first term describes the event in which the current sample and the preceding i samples contain no defective units. The second terms describes the event in which the current sample contains zero defective units, while any of the i preceding samples contains only one defective unit and the rest of the $(i-1)$ samples are free from defective units.

Another type of a dependent sampling plan is a MDS plan. A MDS plan for inspection by attributes is specified by four parameters (n, c_1, c_2, i) and has the following operating procedure:

(1) From each lot, select a sample of n units and observe the number of defective units $D_{n,p}$.
(2) If $D_{n,p} \leq c_1$, accept the lot; If $D_{n,p} > c_2$, reject the lot; If $c_1 < D_{n,p} \leq c_2$, accept the current lot if all the immediately preceding i lots are accepted.

The OC-curve of a MDS-(n, c_1, c_2, i) for inspection by attributes is determined by:

$$\phi_a^{\text{MDS}}(p, n, c_1, c_2, i) = P(D_{n,p} \leq c_1) + P(c_1 < D_{n,p} \leq c_2) \phi_a^{\text{MDS}}(p, n, c_1, c_2, i)^i. \tag{3}$$

If $c_1 = 0$ and $c_2 = 1$, the OC-curve of a MDS plan approximates that of a ChSP-1 plan. Moreover, it is clear from equation (3) that for large i the OC-curve is approximated by that of a SSP-(n, c_1) plan.

If a continuous variable X in a lot of size N is inspected, one can use a SSP plan *for inspection by variables* [13]. Such plan assumes that the variable X follows a normal distribution $N(\mu, \sigma^2)$ with mean μ and a standard deviation σ. In this article, it is assumed that σ is known. If an upper specification limit U is given, a single lot is accepted under the condition $\frac{U-\overline{X}_n}{\sigma} \geq k$, where k is a continuous criterion indicating the minimal standardized distance between the sample mean $\overline{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ and the upper specification limit U. Similarly, if a lower-specification limit L is used, the condition is given by $\frac{\overline{X}_n-L}{\sigma} \geq k$. In both cases the OC-curve for a SSP-(n, k) for variables inspection is defined by:

$$\phi_v(p, n, k) = P(Z \leq \sqrt{n}(\Phi^{-1}(1-p) - k)), \tag{4}$$

where Φ denotes the cumulative distribution function of a variable Z following a standard normal distribution $N(0, 1)$.

Govindaraju [8] provided upper limits $k_{\max}$ on k to decide whether a given SSP-(n, k) has a satisfactory OC-curve or not. A ChSP-1-(n, k, i) for variables inspection is applied when the criterion k of a SSP-(n, k) exceeds $k_{\max}$. When an upper-specification limit U (resp. lower-specification limit L) is defined on the measured variable X, the procedure is as follows:

(1) From each lot, select a sample of n units and compute $V = \frac{U-\overline{X}_n}{\sigma}$ (resp. $V = \frac{\overline{X}_n-L}{\sigma}$).
(2) Accept the lot if $V \geq k_{\max}$ and reject the lot if $V < k_1$. If $k_1 < V < k_{\max}$, the lot is accepted provided the preceeding i lots were accepted on the condition that $v \geq k_{\max}$.

The parameter k_1 is determined by applying the two-point method to the OC-curve that is defined by:

$$\phi_v^{\text{ChSP}}(p, n, k, i) = \phi_v(p, n, k_{\max}) + \Big(\phi_v(p, n, k_1) - \phi_v(p, n, k_{\max})\Big)\phi_v(p, n, k)^i.$$

As with ChSP-1 plan for attributes inspection, the performance of these plans approaches that of a SSP-$(n, k_{\max})$ plan for large i.

The MDS plans for variables inspection follow a similar procedure as the ChSP-1 plans by variables, but where $k_{\max}$ is replaced by another criterion $k_2 > k_1$. The plans are defined by four parameter (n, k_1, k_2, i) and their OC-curve is given by:

$$\phi_v^{\text{MDS}}(p, n, k_1, k_2, i) = \phi_v(p, n, k_2) + \Big(\phi_v(p, n, k_1) - \phi_v(p, n, k_2)\Big)\phi_v^{\text{MDS}}(p, n, k_1, k_2, i)^i.$$

3. Modified chain sampling plans

In this section MChSP plans are defined, first for inspection by attributes and then for inspection by variables. For this purpose, it is assumed that lots are drawn from a

continuing stream of lots of a process with an unknown but constant fraction defective p. Furthermore, the dependency of the OC-curves with respect to the plan parameters is formally studied.

A MChSP plan for inspection by attributes is determined by a triple of natural numbers (n, c, i) and has the following operating procedure:

(1) Select a sample of size n from the current lot and observe the number of defective units $D_{n,p}$. Reject the lot if $D_{n,p} > c$.
(2) If $D_{n,p} \leq c$ the lot is accepted provided that there is at most 1 lot among the preceding i lots in which the number of defective units $D_{n,p}$ exceeds the criterion c. Otherwise reject the lot.

The OC-curve of a MChSP-(n, c, i) plan for inspection by attributes is given by:

$$\phi_a^*(p, n, c, i) = m(m^i + im^{i-1}(1-m)), \tag{5}$$

where $m = P(D_{n,p} \leq c)$ is given by the probability that the observed number of defective units found in a lot is less than the criterion c. The OC-curve of a MChSP-$(n, 0, i)$ plan will approximate that of a MChSP-1-(n, i) plan. Furthermore, for $i = 1$, one retrieves the SSP by attributes, i.e., $\phi_a^*(p, n, c, 1) = m(p, n, c) = \phi_a(p, n, c)$.

In a similar way the results of a SSP plan by variables can be chained to obtain a MChSP by variables. For this purpose a variable $X \sim N(\mu, \sigma)$ is considered with σ known. When an upper-specification limit U (resp. lower-specification limit L) on X is defined, the operating procedure of a MChSP-(n, k, i) plan for inspection by variables is given by:

(1) From the current lot, select a sample of n units and compute $V = \frac{U-\overline{X}_n}{\sigma}$ (resp. $V = \frac{\overline{X}_n-L}{\sigma}$). Reject the lot if $V < k$.
(2) When $V > k$ the lot is accepted provided there is at most 1 lot among the preceding i lots where $V < k$. Otherwise reject the lot.

Thus the OC-curve is given by:

$$\phi_v^*(p, n, k, i) = w(w^i + iw^{i-1}(1-w)), \tag{6}$$

where $w = w(p, n, k) = P(Z \leq \sqrt{n}(\Phi^{-1}(1-p) - k))$.

We proceed by discussing several properties of MChSP plans. Proofs of the theorems can be found in Appendix A. To start with, a property of a SSP-(n, c) plan by attributes is studied. Note that a SSP-(n, c) plan can be viewed as a MChSP-(n, c, i) plan with $i = 1$. The design of SSP plans is based on the solutions of the system of equations (1), which generally do not have to be unique. Geometrically, the solutions will be situated in a region of the nc-plane consisting of all the points (n, c) that satisfy $c_l(n) \leq c \leq c_u(n)$ with

$$c_l(n) = \sup\{c | \phi_a(p_{\text{RQL}}, n, c) \leq \beta\} \quad \text{and} \quad c_u(n) = \inf\{c | \phi_a(p_{\text{AQL}}, c, n) \geq 1-\alpha\}. \tag{7}$$

The expression of a minimum size SSP plan that passes through a consumer's risk point (p_{RQL}, β) is due to Hahn [12]. The following theorem shows that these plans are obtained as solutions of (1) as $p_{\text{AQL}} \to 0$.

Theorem 3.1. *Consider a consumer's risk point (p_{RQL}, β) and producer's risk α. As $p_{\text{AQL}} \to 0$, the sampling plan (n, c) satisfying (1) that has the smallest sample size is*

given by the zero-acceptance number sampling plan:

$$\left(\left\lceil \frac{\log \beta}{\log(1-p_2)} \right\rceil, 0\right).$$

Thus, for high yield processes where the defect ratio p_{AQL} is very low, one tends to find zero-acceptance number sampling plans as solutions of Eq. (1). As noted in Section 2, these plans are undesirable for the producer as their OC-curves tend to decrease very rapidly near zero. MChSP plans ensure a discriminative power for small fractions defective by the existence of an inflection point on the OC-curve, as is shown in the next theorem.

Theorem 3.2. *Consider a MChSP-(n,c,i) plan and its corresponding OC-curve $p \mapsto \phi_a^*(p,n,c,i)$. Then:*

(1) The OC-curve $p \mapsto \phi_a^*(p,n,c,1)$ of a SSP-(n,c) plan is situated above the OC-curve $p \mapsto \phi_a^*(p,n,c,i)$ of a MChSP-(n,c,i) plan with $i > 1$.
(2) $\phi_a^*(p,n,c,i)$ is decreasing as a function of n,i and increasing as a function of c.
(3) If $i > 1$, there exists an inflection point $p = p_1$, where $\frac{\partial^2 \phi_a^*}{\partial p^2}(p_1,n,c,i) = 0$.

Figure 1(a) shows OC-curves of several MChSP plans for inspection by attributes. As n or i increases the curves become steeper such that p_{AQL} and p_{RQL} can both be chosen smaller. Furthermore, the OC-curves possess a point of inflection which tends to move closer to the y-axis as i or n increases.

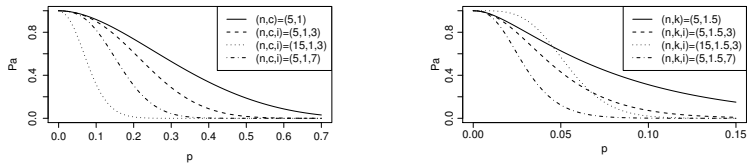
The next theorem deals with SSP plans for inspection by variables.

Theorem 3.3. *Consider a SSP-(n,k) plan by variables. Denote $p_0 = 1-\Phi(k)$, then:*

(1) $\phi_v(p,n,k)$ decreases as a function of k.
(2) $\phi_v(p,n,k)$ increases as a function of n for $p < p_0$ and decreases as a function of n for $p > p_0$.

Furthermore $\phi_v(p_0,n,k) = \frac{1}{2}$.

From theorem 3.3, it is clear that an increase in the sample size of a SSP plan will result in both a decrease of the producer's risk for $p_{\mathrm{AQL}} < p_0$ and a decrease in the consumer's risk for $p_{\mathrm{RQL}} > p_0$. This is not the case for SSP plans by attributes, where an increase in sample size will only decrease the consumer's risk (as is shown



(a) SSP-(n,c) and MChSP-(n,c,i) plans for attributes inspection.

(b) SSP-(n,k) and MChSP-(n,k,i) plans for variables inspection.

Figure 1. OC-curves of SSP and MChSP plans showing the lot acceptance probability P_a as a function of the lot fraction defective p.

in Theorem 3.2). The quality level at p_0 where $\phi_v(p_0,n,k) = \frac{1}{2}$ is also termed the *indifference quality level* [5, 21].

Theorem 3.4. *Consider a MChSP-(n,k,i) plan by variables. Denote $p_0 = 1-\Phi(k)$, then:*

(1) $\phi_v^*(p,n,k,i)$ decreases as a function of i and k.
(2) $\phi_v^*(p,n,k,i)$ increases as a function of n for $p < p_0$ and decreases as a function of n for $p > p_0$.
(3) There exists an inflection point $p = p_1$, where $\frac{\partial^2 \phi_v^*}{\partial p^2}(p_1,n,k,i) = 0$.

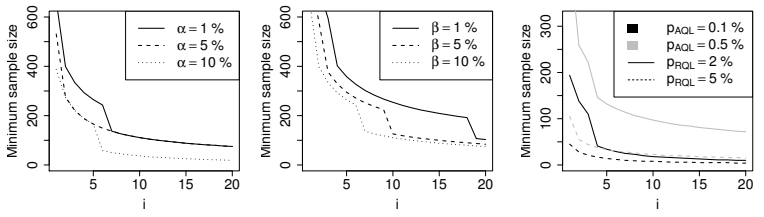
Furthermore $\phi_v^*(p_0,n,k,i) = \frac{i+1}{2^{i+1}}$.

Figure 1(b) shows OC-curves of several MChSP plans by variables. In contrast to the MChSPs for inspection by attributes, the discriminative power of the plans improves as n increases (as shown in Figure 1(a)). The curves become steeper when the number of chained sample results i increases.

4. Statistical design and parameter sensitivity analysis

In this section, search procedures are discussed to solve the system of equations (1) for MChSP plans. Furthermore, the effect of the choice of the parameters α, β, p_{AQL} and p_{RQL} on the required sample sizes is studied.

Appendix B provides the pseudocode for procedures to search a series of solutions of the system of equations (1). The search strategies calculate for each i the minimum sample size n_i^{min} that is required to assure that the OC-curve of the sampling plan will pass approximately through a given producer's point $(p_{\mathrm{AQL}}, 1-\alpha)$ and a given consumer's point $(p_{\mathrm{RQL}}, \beta)$. Moreover, for each i, corresponding criteria c_i^{min} and k_i^{min} are calculated for inspection by attributes and variables respectively. In this way a solution curve $\mathcal{S} : n = n(i)$ is found that depends on the chosen quality levels p_{AQL}, p_{RQL} and risks α, β. Note that a similar search procedure can be used to select MChSP-1 plans that are based on binomial models. Technical details can be found in Appendix B.

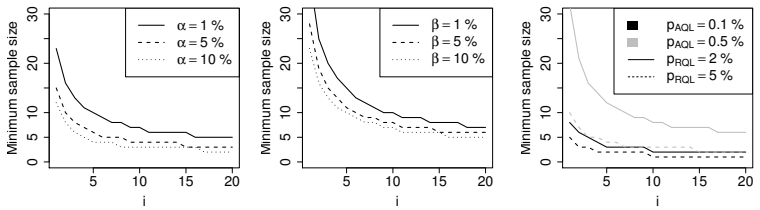


(a) Effect of changing α.

(b) Effect of changing β.

(c) Effect of changing p_{AQL} and p_{RQL}.

Figure 2. Effect of changing risks (α,β) and changing quality levels $(p_{\mathrm{AQL}}, p_{\mathrm{RQL}})$ on the solution curves of MChSP plans for attributive inspection. One parameter at a time is changed starting from the point $(\alpha,\beta,p_{\mathrm{AQL}},p_{\mathrm{RQL}}) = (5\%,10\%,0.1\%,1\%)$ in parameter space.



(a) Effect of changing α. (b) Effect of changing β. (c) Effect of changing p_{AQL} and p_{RQL}.

Figure 3. Effect of changing risks (α, β) and changing quality levels $(p_{\mathrm{AQL}}, p_{\mathrm{RQL}})$ on the solution curves of MChSP plans for variable inspection. One parameter at a time is changed starting from the point $(\alpha, \beta, p_{\mathrm{AQL}}, p_{\mathrm{RQL}}) = (5\%, 10\%, 0.1\%, 1\%)$ in parameter space.

Figure 2 and Figure 3 show solution curves of MChSP plans for different values of the parameters $\alpha, \beta, p_{\mathrm{AQL}}$ and p_{RQL}. The solution curves may show sharp corners and jumps due to the discreteness of the sample sizes. To examine the effect of the choice of parameters on the solutions curves, one parameter at a time is changed [16]. For illustrating purposes, the risks are set to $\alpha = 5\%$ and $\beta = 10\%$, which are common choices. The acceptance quality level p_{AQL} is set to 0.1%, and the rejectable quality level p_{RQL} is set to 1%. Figure 2(a) and 3(a) show the effect on the solution curves of a change in α; Figure 2(b) and 3(b) show the effect on the solution curves of a change in β. Generally an increase in risk naturally leads to a decrease in sample size. Note that, the underlying criteria c_i^{min} vary between 0 and 4 and tend to decrease as well with increasing risks. Furthermore, lower sample sizes are naturally accompanied by lower criteria.

Figure 2(c) and 3(c) show the effect of changes in p_{AQL} and p_{RQL}. It is clear that an increase of p_{AQL} while p_{RQL} is held constant leads to an increase in sample size. Similar an increase in sample size is expected when p_{RQL} decreases while p_{AQL} is held constant. Generally, it is expected that a reduction in distance between p_{RQL} and p_{AQL} leads to an increase in the sample size (and an increase in the accompanied criteria) as this imposes a higher steepness to the OC-curves.

5. Comparison between sampling plans

In this section, the sampling efficiency is compared between MChSP plans and other dependent sampling procedures. In particular, the required sample size of MChSP plans is compared with that of several sampling procedures: (i) ChSP-1 plans [11]; (ii) SSP plans [17]; (iii) MDS plans[1, 20], and (iv) MChSP-1 plans for inspection by attributes based on a binomial model [8].

Table 1 shows a comparison between different sampling plans by attributes that are designed by using the two-point approach with different values of p_{AQL} and p_{RQL}. The risks α at p_{AQL} and β at p_{RQL} are limited to 5% and 10%, respectively. Sample sizes are minimized when computing ChSP-1 and MDS plans using the procedures in [17, 20]. The design of MChSP-1 and MChSP plans is based on the algorithms discussed in Appendix B, and the same number of chained samples i is chosen as in

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Table 1. Several attributes sampling plans indexed by p_{AQL} and p_{RQL}: ChSP-1-(n, i) plans, MDS-(n, c_1, c_2, i), MChSP-1-(n, i), MChSP$-(n, c, i)$, and MChSP$-(n, c, i')$ plans, where i' is given by the minimum number of past lot results that are required to obtain a smaller sample size than the matching MChSP-1 plan and MDS plan. Risks are set to $\alpha = 5\%$ and $\beta = 10\%$.

p_{AQL}	p_{RQL}	ChSP-1		MDS				MChSP-1		MChSP					
		n	i	n	c_1	c_2	i	n	i	n	c	i	n	c	i'
0.001	0.008	/	/	**287**	0	2	4	/	/	**235**	1	4			
	0.01	/	/	**230**	0	1	3	/	/	**221**	1	3			
	0.02	**114**	4	**114**	0	1	4	**37**	4	**41**	0	4	**34**	0	5
	0.05	**45**	3	**45**	0	1	3	**18**	3	**21**	0	3	**17**	0	4
0.002	0.01	/	/	**235**	0	2	2	/	/	**339**	2	2	**137**	1	7
	0.02	/	/	**115**	0	1	3	/	/	**110**	1	3			
	0.05	**45**	3	**45**	0	1	3	**18**	3	**21**	0	3	**17**	0	4
	0.1	**22**	3	**22**	0	1	3	**9**	3	**10**	0	3	**8**	0	4
0.005	0.02	/	/	**118**	0	3	2	/	/	**259**	3	2	**114**	2	7
	0.05	/	/	**45**	0	1	3	/	/	**44**	1	3			
	0.1	**22**	3	**22**	0	1	3	/	/	**22**	1	3	**8**	0	4
	0.2	**11**	2	**11**	0	1	2	**6**	2	**7**	0	2	**5**	0	3
0.01	0.05	/	/	**46**	0	2	2	/	/	**79**	2	2	**26**	1	8
	0.1	/	/	**22**	0	1	3	/	/	**22**	1	3	**19**	1	4
	0.2	**11**	2	**11**	0	1	2	**6**	2	**13**	1	2	**4**	0	4
	0.3	**7**	2	**7**	0	1	2	**4**	2	**9**	1	2	**3**	0	3

Table 2. Relative sampling efficiencies $\frac{n_0}{n}$ with respect to a SSP-(n_0, c) plan for several attributes sampling plans: ChSP-1-(n, i), MDS-(n, c_1, c_2, i), MChSP-1-(n, i), MChSP-(n, c, i), and MChSP-(n, c, i') plans with i' as in Table 1. Risks are set to $\alpha = 5\%$ and $\beta = 10\%$.

p_{AQL}		0.001				0.002				0.005				0.01			
p_{RQL}		0.008	0.01	0.02	0.05	0.01	0.02	0.05	0.1	0.02	0.05	0.1	0.2	0.05	0.1	0.2	0.3
ChSP-1				1.7	1			1.7	1			1.7	1.6			1.6	1.7
MDS		2.3	2.3	1.7	1	2.8	2.3	1.7	1	3.9	2.3	1.7	1.6	2.9	2.4	1.6	1.7
MChSP-1				5.2	2.5			4.2	2.4				3			4.5	3
MChSP		2.8	2.4	4.7	2.1	2.0	2.4	3.7	2.2	1.8	2.4	1.7	2.6	1.7	2.4	1.4	1.3
MChSP	$i = i'$			5.7	2.6	4.9		4.5	2.8	4.1		4.8	3.6	5.1	2.7	1.6	4
SSP	n_0	664	531	194	45	667	265	77	22	462	105	38	18	132	52	18	12
	c	2	2	1	0	3	2	1	0	5	2	1	1	3	2	1	1

the matching MDS plans.

It can be seen from Table 1 that, when the same number of past sample results is used, MChSP plans do not always require a smaller sample size compared to MDS, ChSP-1 and MChSP-1 plans. However, in contrast to ChSP-1 and MDS plans, the sample sizes of MChSP plans can be further reduced by increasing the number of chained sample results. Table 1 shows the minimum number i' of past sample results that is required to obtain a MChSP-(n, c, i') plan with a smaller sample size than a MDS-(n, c_1, c_2, i) plan and a MChSP-1-(n, c, i) plan. Furthermore, MChSP plans have the appealing ability to possess steeper OC-curves in comparison with MChSP-1 plans, because they allow criteria $c > 0$, i.e., no solution exists for ChSP-1 and MChSP-1 plans if the required distance between p_{AQL} and p_{RQL} is too small.

Table 2 shows a comparison between the sampling efficiencies of the different sampling plans with respect to a SSP-(n_0, c) plan. The quality levels p_{AQL} and p_{RQL} are chosen as in Table 1 and risks α and β are set to 5% and 10% respectively. The relative efficiencies are calculated as the ratios $\frac{n}{n_0}$ of the sample sizes of matching sampling plans. If a sufficient number of past lot results is available the highest sampling efficiency is obtained by MChSP plans. Note that the required sample size of

Table 3. Several variables sampling plans indexed by p_{AQL} and p_{RQL}: ChSP-1-(n, k_1, i), MDS-(n, k_1, k_2, i), MChSP$-(n, k, i)$, and MChSP$-(n, k, i')$ plans, where i' is the minimum number of past lot results that are required to obtain a smaller sample size than the matching MDS plan. Risks are set to $\alpha = 5\%$ and $\beta = 10\%$.

p_{AQL}	p_{RQL}	ChSP-1			MDS				MChSP					
(PPM)	(PPM)	n	k	i	n	k_1	k_2	i	n	k	i	n	k	i'
1	2	/	/	/	**262**	2.06	4.69	2	**281**	4.65	2	**215**	4.64	3
	5	**47**	4.61	2	**47**	2.16	4.61	2	**51**	4.52	2	**39**	4.48	3
	8	**27**	4.57	2	**27**	4.14	4.57	2	**30**	4.45	2	**23**	4.40	3
	10	**22**	4.54	2	**22**	4.07	4.55	2	**24**	4.41	2	**19**	4.36	3
2	5	/	/	/	**127**	1.97	4.55	1	**228**	4.50	1	**115**	4.45	3
	8	/	/	/	**60**	4.30	4.51	1	**98**	4.45	1	**50**	4.37	3
	10	**44**	4.46	2	**44**	4.25	4.49	1	**72**	4.41	1	**37**	4.33	3
	20	**21**	4.40	1	**21**	3.91	4.39	2	**23**	4.26	2	**18**	4.21	3
5	8	/	/	/	**320**	3.40	4.39	1	**812**	4.35	1	**295**	4.31	5
	10	/	/	/	**228**	1.01	4.35	2	**244**	4.31	2	**187**	4.29	3
	20	/	/	/	**55**	3.99	4.29	2	**59**	4.20	2	**46**	4.16	3
	30	**32**	4.24	2	**32**	3.99	4.28	1	**53**	4.19	1	**27**	4.09	3
10	20	/	/	/	**142**	2.74	4.20	2	**229**	4.15	2	**126**	4.10	5
	30	/	/	/	**83**	3.92	4.16	2	**90**	4.09	2	**69**	4.06	3
	40	/	/	/	**51**	3.82	4.12	2	**56**	4.04	2	**43**	4.01	3
	50	**38**	4.14	1	**38**	3.88	4.13	1	**62**	4.01	1	**31**	3.95	3

Table 4. Relative sampling efficiencies $\frac{n_0}{n}$ with respect to a SSP-(n_0, k) plan for several variables sampling plans: ChSP-1-(n, k_1, i), MDS-(n, k_1, k_2, i), MChSP-(n, k, i), and MChSP-(n, k, i') plans with i' as in Table 3. Risks are set to $\alpha = 5\%$ and $\beta = 10\%$.

p_{AQL} (PPM)		1				2				5				10			
p_{RQL} (PPM)		2	5	8	10	5	8	10	20	8	10	20	30	20	30	40	50
ChSP-1			1.6	1.7	1.6			1.6	1.6				1.7				1.6
MDS		1.6	1.6	1.7	1.6	1.8	1.6	1.6	1.6	2.5	1.6	1.6	1.7	2.4	1.6	1.6	1.6
MChSP		1.5	1.5	1.5	1.5	1.0	1.0	1.0	1.5	1.0	1.5	1.5	1.0	1.5	1.5	1.5	1.5
MChSP	$i = i'$	2.0	1.9	2.0	1.9	2.0	2.0	1.9	1.9	2.8	2.0	2.0	2.0	2.7	2.0	2.0	2.0
SSP	n_0	425	76	45	36	228	98	72	34	812	370	90	53	346	135	84	62
	k	4.7	4.6	4.5	4.5	4.5	4.4	4.1	4.3	4.4	4.3	4.2	4.2	4.2	4.1	4.1	4.1

a zero-acceptance SSP plan by attributes equals that of a ChSP-1 plan such that an efficiency of 1 is obtained.

Table 4 shows a comparison between different sampling plans by variables. As before, the plans are selected by using the two-point method. Sample sizes are minimized when computing the ChSP-1 and MDS plans by using the procedures in [1, 17]. MChSP plans are selected by means of Algorithm 2 in Appendix B. Compared to attributes inspection, variables inspection is able to limit the risks α and β at lower proportions defective [19]. Therefore, small quality levels p_{AQL} and p_{RQL} are considered expressed in parts per million (PPM) that may result from high quality processes [23].

The sampling efficiency of a MChSP plan is lower compared to that of a ChSP-1 and a MDS plan when the same number of past sample results is used. If $i = 1$, MChSP plans are equivalent to SSP plans and the difference in efficiency between the plans is largest. However, the sample sizes of MChSP plans rapidly decrease when the number of chained samples i increases. Table 3 shows the minimal number i' of past results that is required to obtain a MChSP-(n, c, i') plan with a smaller sample size n than the matching MDS plan.

Table 4 shows the relative efficiencies of the sampling plans in Table 3 with respect to a matching SSP-(n, k) by variables. The sampling efficiency of MChSP-(n, k, i) plans equals that of a SSP-(n, k) plan if $i = 1$. As i increases, the relative efficiency of a

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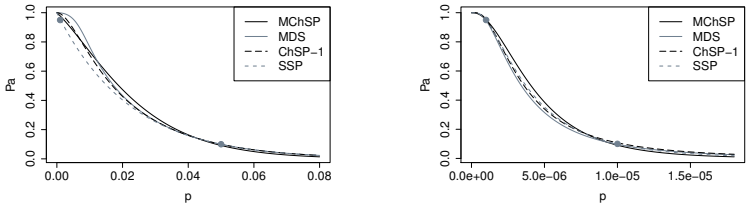


(a) OC-curves for attributive inspection.

(b) OC-curves for variable inspection.

Figure 4. Comparison of matching OC-curves passing approximately through two designated points indicated by the gray dot. (a) Attributes sampling plans: MChSP-$(21, 0, 3)$, MDS-$(45, 0, 1, 3)$, ChSP-1-$(45, 3)$ and SSP-$(45, 0)$. (b) Variables sampling plans: MChSP-$(19, 4.36, 3)$, MDS-$(22, 4.07, 4.55, 2)$, ChSP-1-$(22, 4.54, 2)$ and SSP-$(36, 4.48)$.

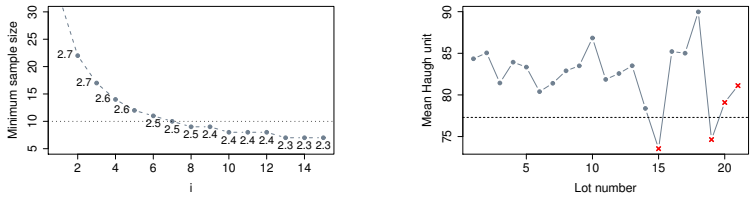
MChSP plans with respect to a SSP increases.

Finally, Figure 4 shows the OC-curves of several matching sampling plans that pass (approximately) through two designated points: $(p_{\mathrm{AQL}}, 1-\alpha)$ and $(p_{\mathrm{RQL}}, \beta)$ (indicated by a gray dot). For attributes sampling plans , the parameters were chosen to be $p_{\mathrm{AQL}} = 0.001$, $p_{\mathrm{RQL}} = 0.05$, $\alpha = 0.05$, and $\beta = 0.1$ (as shown in Figure 4(a) and Table 1); For variables sampling plans, the parameters were chosen to be $p_{\mathrm{AQL}} = 1$ PPM, $p_{\mathrm{RQL}} = 10$ PPM, $\alpha = 0.05$, and $\beta = 0.1$ (as shown in Figure 4(b) and Table 3). As noted before, the lot acceptance probability of a zero-acceptance sampling plan begins to drop very rapidly as the lot fraction defective becomes greater than p_{AQL}. The MDS plans have a high discriminative power, ensuring producer's protection, even for fractions defective beyond p_{AQL}. The OC-curves of the MChSP and ChSP-1 plan are situated between those of the matching MDS and SSP plan. However, the discriminative power of the ChSP-1 plans is slightly higher when compared to the MChSP plans.

6. Example

To show how the sampling plan can be applied to a real-world scenario, a case study from food industry is presented as an example. A company that is specialized in egg processing inspects shipments of eggs from a local farmer. The quality characteristic that is inspected is given by the Haugh unit, which is a measure of egg protein quality based on the height of its egg white.

Suppose that the lower specification limit for the Haugh unit is set to $L = 65$ and the company desires to restrict the required sample size to 10 when inspecting a random sample from a lot. One assumes that the lot fraction defective is constant and the standard deviation of the Haugh unit is known as $\sigma = 5.0$. The company and the farmer agree to set the quality levels p_{AQL} and p_{RQL} as 0.1% and 0.5% respectively. The risks α and β are set to 5% and 1% respectively. Figure 5(a) shows the solution curve of the required sample sizes as a function of the chained sample results i. The company decides to use a number of $i = 8$ past lot results with a sample size of $n = 9$ and a criterion $k = 2.46$.



(a) (b)

Figure 5. (a) Required sample size as a function of past lot results associated with the parameters $p_{\text{AQL}} = 0.1\%$, $p_{\text{RQL}} = 0.5\%$, $\alpha = 5\%$, and $\beta = 1\%$. The labels indicate the criteria. (b) Sample means of the past 21 shipments for the Haugh unit case study. The crosses indicate that the lots are rejected.

In Figure 5(b), the mean Haugh unit of the last 21 samples of size $n = 9$ are shown. The dashed line indicates the critical distance $k\sigma = 12.3$ from the lower specification limit $L = 65$. Starting from lot 9, the MChSP plan can be applied using the past 8 lot results. The first 14 lot results are accepted under the MChSP plan. The mean Haugh unit of lot 19 is too close to the lower specification limit of the Haugh unit such that it is rejected. Lots 16 through 19 are accepted because the preceding 8 lots contain at most 1 lot (lot 15 in particular) that was rejected according to the SSP-$(9, 2.46)$ plan by variables. Also, lot 19 is rejected. Lot 20 and 21 are rejected as the preceding 8 lots contain two lots with a sample mean that is too close to L.

7. Conclusion

In this article MChSP plans were introduced that are an extension of the existing ChSP-1 and MChSP-1 plans. Compared to ChSP-1 plans, MChSP plans enable us to reduce the sample size by increasing the number of past lot results that are considered during inspection. Several properties of MChSP plans were derived to study the dependency of their OC-curve as a function of the sample size n, the number of chained samples i and the criterion. Search procedures were developed to select MChSP plans with an OC-curve that passes through two designated points: the producer's risk point $(p_{\text{AQL}}, 1 - \alpha)$ and the consumer's risk point (p_{RQL}, β).

The proposed plans were compared to ChSP-1, MDS and SSP plans in terms of sampling efficiency. It is shown that the required sample size of a MChSP plan is smaller than that of a matching ChSP-1, MDS or SSP plan if a sufficient number of past sample results is available. Furthermore, MChSP plans have the ability to possess steeper OC-curves compared to ChSP-1 and MChSP-1 plans such that p_{AQL} and p_{RQL} can both be chosen smaller when using the two-point method for designing the plans.

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Appendix A. Proofs of theorems

In this appendix the proofs of the theorems that were presented in Section 3 are given.

A.1. *Proof of Theorem 3.1*

Let $n_0 = \left\lfloor \frac{\log(1-\alpha)}{\log(1-p_{\mathrm{AQL}})} \right\rfloor$ for a given producer's risk point $(p_{\mathrm{AQL}}, \alpha)$. Then:

$$\forall n \leq n_0 : \phi_a(p_{\mathrm{AQL}}, 0, n) \quad = \quad (1 - p_{\mathrm{AQL}})^n \geq 1 - \alpha$$

such that $\forall n \leq n_0 : c_u(n) = 0$. Therefore, for small p_{AQL}, the desired sampling plans have a criterion $c = 0$ upto the sample size $n_0 = \left\lfloor \frac{\log(1-\alpha)}{\log(1-p_{\mathrm{AQL}})} \right\rfloor$. Clearly $n_0 \to +\infty$ as $p_{\mathrm{AQL}} \to 0$. Therefore, given a consumer's risk point $(p_{\mathrm{RQL}}, \beta)$ and producer's risk α, one finds a $\delta > 0$ such that:

$$\forall p_{\mathrm{AQL}} < \delta : n_0 \geq \frac{\log \beta}{\log(1 - p_{\mathrm{RQL}})}$$

and thus $(1 - p_{\mathrm{RQL}})^{n_0} \leq \beta$. Hence, for $p_{\mathrm{AQL}} < \delta$ the SSP by attributes with the lowest sample size that satisfies the requirements from as well producer's as consumer's side is given by $(n, c) = \left(\left\lceil \frac{\log \beta}{\log(1-p_{\mathrm{RQL}})} \right\rceil, 0 \right)$.

A.2. *Proof of Theorem 3.2*

Firstly, remark that the function $P_a = \phi_a^*(p, n, c, i) = m(m^i + im^{i-1}(1 - m))$ can be extended to allow positive real values for m and i. This enables us to consider partial derivatives with respect to m and i.

(i) Clearly, $0 \leq m^i + im^{i-1}(1 - m) \leq (m + (1 - m))^i = 1$, for each $i \geq 1$, such that $0 \leq \phi_a^*(p) \leq m$.
(ii) The function $p \mapsto \phi_a^*(p, n, c, i)$ is increasing as a function of $m \in]0, 1[$:

$$\frac{\partial \phi_a^*}{\partial m} = m^i + i^2 m^{i-1}(1 - m) > 0$$

such that ϕ_a^* has the same monotonicity as m with respect to n and c. As the function m represents the acceptance under a SSP-(n, c) plan, it is clear that ϕ_a^* decreases as a function of n and increases as a function of c. The monotonicity as a function of $i > 1$